DYNAMICAL WEYL GROUPS AND APPLICATIONS

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Abstract. Following a preceding paper of Tarasov and the second author, we define and study a new structure, which may be regarded as the dynamical analogue of the Weyl group for Lie algebras and of the quantum Weyl group for quantized enveloping algebras. We give some applications of this new structure.

1. Introduction

In 1994, G.Felder, in his pioneering work [F], initiated the development of a new area of the theory of quantum groups – the theory of dynamical quantum groups. This theory assigns dynamical analogs to various objects related to ordinary Lie algebras and quantum groups (e.g. Hopf algebras, R-matrices, twists, etc.) In particular, the main goal of the present paper is to assign a dynamical analog to the Weyl group of a Kac-Moody Lie algebra $\mathfrak{g}$ and the quantum Weyl group of the corresponding quantum group. More specifically, we give a (rather straightforward) generalization of the main construction of the paper [TV], which, in effect, introduces dynamical Weyl groups in the case of finite dimensional simple Lie algebras.

The analog of the Weyl group we introduce is a collection of operators that give rise to a braid group representation on the space of functions from the dualized Cartan subalgebra $\mathfrak{h}^*$ of $\mathfrak{g}$ to a representation V of $\mathfrak{g}$ or $U_q(\mathfrak{g})$. We call this analog **the dynamical Weyl group** of V.

We note that dynamical Weyl groups may be regarded as generalizations of the classical "extremal projectors" introduced in [AST]. In particular, the dynamical Weyl group operators for simple Lie algebras were introduced in [Zh1, Zh2], by analogy with [AST]

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(see formula (3.5) and Theorem 2 in [Zh2]). This construction, however, is different from that of [TV].

Dynamical Weyl groups are not only beautiful objects by themselves, but also have a number of useful applications. To describe one of these applications, recall that in [EV2], we developed the theory of trace functions (matrix analogs of Macdonald functions), using the basic dynamical objects introduced in [EV1] (the fusion and exchange matrices). In particular, we derived four systems of difference equations for these functions: qKZB, dual qKZB, Macdonald-Ruijsenaars, and dual Macdonald-Ruijsenaars equations, and proved the symmetry of the trace functions under permutation of components and arguments simultaneously. In this paper, we use the dynamical Weyl group to develop this theory further: namely, we show that the trace functions and all four systems of equations for them are symmetric with respect to the dynamical Weyl group (while they are not, in general, symmetric under the usual classical or quantum Weyl group). This property is a generalization to the matrix case of the Weyl symmetry property of Macdonald functions; being important by itself, it also allows one to prove other properties of trace functions (orthogonality, the Cherednik-Macondald-Mehta identities), which we plan to do in a separate paper.

As a second application, we interpret the important operator $Q(\lambda)$ from [EV2] in terms of the dynamical Weyl group operator corresponding to the maximal element of the Weyl group. This allows us to calculate $Q(\lambda)$ explicitly, and in particular get an explicit product formula for its determinant. In the next paper, we will show that $Q(\lambda)$ is the matrix analog of the squared norm of the Macdonald polynomial P_λ, and in particular the product formula for the determinant of $Q(\lambda)$ is the matrix analog of the well known Macdonald inner product identities.

Finally, a third application of dynamical Weyl groups is to the theory of KZ and qKZ equations, and is along the lines of [TV]. Recall that the main goal of [TV] was not to define dynamical Weyl groups for simple Lie algebras, but rather to construct commuting difference operators which commute with the trigonometric KZ operators and are a deformation of the differential operators from [FMTV] commuting with the KZ operators. Such difference operators were constructed in [TV] using the dynamical Weyl group of the corresponding simple Lie algebra, combined with the method of Cherednik [Ch1] of lifting R-matrices to affine R-matrices. However, the method of [TV] allowed the authors of [TV] to prove that their operators actually commute with the trigonometric KZ operators only for Lie algebras other than E_8, F_4, G_2 (i.e. for Lie algebras having a minuscule fundamental coweight). In this paper, we attack the same problem using a somewhat different method: we use the dynamical Weyl group of the *affine* Lie algebra, rather than the finite dimensional one, which allows us to avoid using the procedure from [Ch1]. As a result, we obtain the same difference operators as in [TV], and prove that they commute with the KZ operators for any simple Lie algebra $\mathfrak{g}$. We also generalize the constructions and results of [TV] to the quantum group case.

We note that another unexpected application of dynamcial Weyl groups appears in the recent interesting paper [RS].

The organization of the paper is as follows.

In Section 2, we recall the basic notions used in this paper: Kac-Moody algebras and their quantizations, (ordinary) Weyl groups, intertwining operators, Verma modules, singular vectors, fusion and exchange matrices.

In Section 3, using the operation of restriction of intertwining operators for Verma modules to their submodules, we define the dynamical Weyl group operator for $U_q(\mathfrak{s}l_2)$. We calculate this operator explicitly.

In Section 4, we use the operator from Section 3 to define the dynamicl Weyl group operators for any quantized Kac-Moody algebra. We show that similarly to the $U_q(\mathfrak{s}l_2)$ case, the dynamical Weyl group arises from the restriction procedure for intertwiners.

In Section 5, we continue to study the properties of the dynamical Weyl group, and, in particular, show that its limit at infinity gives the usual quantum Weyl group of Soibelman and Lusztig.

In Section 6 we give the first application of the dynamical Weyl group: we link the operator $Q(\lambda)$ from [EV2] with the dynamical Weyl group operator corresponding to the maximal element of the Weyl group.

In Section 7, we describe the applications of the dynamical Weyl group to trace functions. We establish the dynamical Weyl group symmetry of these functions and of the equations for them introduced in [EV2].

In Section 8, we discuss the dynamical Weyl groups of loop representations for affine Lie algebras and quantum affine algebras.

In Section 9, using the material of Section 8, we give a more conceptual derivation of the difference equations from [TV], which commute with the trigonometric KZ equations. We also derive the q-analogs of the equations from [TV], which commute with the trigonometric quantum KZ equations.

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2. Preliminaries

2.1. **Kac-Moody algebras.** We recall definitions from [K]. Let $A = (a_{ij})$ be a symmetrizable generalized Cartan matrix of size r, and $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. This means that $\mathfrak{h}$ is a vector space of dimension $2r - \mathrm{rank}(A)$, $\Pi = \{\alpha_1, ..., \alpha_r\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{h_1, ..., h_r\} \subset \mathfrak{h}$ are linearly independent, and $\alpha_i(h_j) = a_{ji}$. The elements α_i are called simple positive roots.

Definition: The Kac-Moody Lie algebra $\mathfrak{g}(A)$ is generated by $\mathfrak{h}, e_1, ..., e_r, f_1, ..., f_r$ with defining relations

$$[h, h'] = 0,\ h, h' \in \mathfrak{h};\ [h, e_i] = \alpha_i(h)e_i;\ [h, f_i] = -\alpha_i(h)f_i;\ [e_i, f_j] = \delta_{ij}h_i,$$

and the Serre relations

$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{m!(1-a_{ij}-m)!} e_i^{1-a_{ij}-m} e_j e_i^m = 0,$$

$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{m!(1-a_{ij}-m)!} f_i^{1-a_{ij}-m} f_j f_i^m = 0.$$

For brevity we will assume that A is fixed and denote $\mathfrak{g}(A)$ simply by $\mathfrak{g}$. The positive and negative nilpotent subalgebras of $\mathfrak{g}$ will be denoted by $\mathfrak{n}_\pm$.

By the definition of a generalized Cartan matrix, there exists a collection of positive integers d_i, $i = 1, ..., r$, such that $d_i a_{ij} = d_j a_{ji}$. We will choose the minimal collection of such numbers, i.e. the collection for which the numbers are the smallest possible (such a choice is unique). Let us choose a nondegenerate bilinear symmetric form on $\mathfrak{h}$ such that $(h, h_i) = d_i^{-1}\alpha_i(h)$. It is easy to see that such a form always exists. It is known [K] that there exists a unique extension of the form $(,)$ to an invariant nondegenerate symmetric bilinear form $(,)$ on $\mathfrak{g}$. For this extension, one has $(e_i, f_j) = \delta_{ij} d_i^{-1}$.

Remark. One can show that forms on $\mathfrak{g}$ coming from different forms on $\mathfrak{h}$ are equivalent under automorphisms of $\mathfrak{g}$.

A root of $\mathfrak{g}$ is a nonzero element of $\mathfrak{h}^*$ which occurs in the decomposition of $\mathfrak{g}$ as an $\mathfrak{h}$-module. A root is positive if it is a positive linear combination of simple positive roots, and negative otherwise. A root α is real if $(\alpha, \alpha) > 0$, otherwise it is imaginary. For a real root α of $\mathfrak{g}$, let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the corresponding coroot.

2.2. **Quantized Kac-Moody algebras.** Let $\hbar$ be a complex number, which is not a rational multiple of πi, and $q = e^{\hbar/2}$. For a number or operator B, by q^B we mean $e^{\hbar B/2}$.
Definition: The quantized Kac-Moody algebra $U_q(\mathfrak{g}(A))$ is the associative algebra generated by $e_1, ..., e_r, f_1, ..., f_r$, and $q^h, h \in \mathfrak{h}$ (where $q^0 = 1$), with defining relations

$$q^h q^{h'} = q^{h+h'},\ h, h' \in \mathfrak{h};\ q^h e_i = q^{\alpha_i(h)} e_i q^h;\ q^h f_i = q^{-\alpha_i(h)} f_i q^h;\ [e_i, f_j] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}},$$

and the Serre relations

$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_{q_i}![1-a_{ij}-m]_{q_i}!} e_i^{1-a_{ij}-m} e_j e_i^m = 0.$$

$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_{q_i}![1-a_{ij}-m]_{q_i}!} f_i^{1-a_{ij}-m} f_j f_i^m = 0,$$

where $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$, and $q_i := q^{d_i}$.

For brevity we will denote $U_q(\mathfrak{g}(A))$ by $U_q(\mathfrak{g})$. Also, to give a uniform treatment of the classical and quantum case, we will often allow $\hbar$ to be 0 (i.e. $q = 1$), in which case $U_q(\mathfrak{g})$ is defined to be $U(\mathfrak{g})$.

The positive and negative nilpotent subalgebras in $U_q(\mathfrak{g})$ will be denoted by $U_q(\mathfrak{n}_\pm)$.

The algebra $U_q(\mathfrak{g})$ is a Hopf algebra, with coproduct defined by

$$\Delta(q^h) = q^h \otimes q^h, \Delta(e_i) = e_i \otimes q_i^{h_i} + 1 \otimes e_i, \Delta(f_i) = f_i \otimes 1 + q_i^{-h_i} \otimes f_i,$$

and the antipode defined by

$$S(e_i) = -e_i q_i^{-h_i},\ S(f_i) = -q_i^{h_i} f_i,\ S(q^h) = q^{-h}.$$

2.3. **Verma modules and integrable modules.** Let $\lambda \in \mathfrak{h}^*$ be a weight. We say that a vector v in a module V over $\mathfrak{g}$ or $U_q(\mathfrak{g})$ has weight λ if $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$ (respectively $q^h v = q^{\lambda(h)}v$). The space of vectors of weight λ is denoted by $V[\lambda]$. Modules in which any vector is a sum of vectors of some weight are said to be $\mathfrak{h}$-diagonalizable. Category $\mathcal{O}$ consists of $\mathfrak{h}$-diagonalizable modules with finite dimensional weight subspaces, whose weights belong to a union of finitely many "conical" sets of the form $\lambda - \sum_i \mathbb{Z}_+\alpha_i$.

An example of an object in $\mathcal{O}$ is a Verma module. The Verma module M_λ over $\mathfrak{g}$ with highest weight λ is generated by one generator v_λ with defining relations $e_i v_\lambda = 0$, $hv_\lambda = \lambda(h)v_\lambda$, $h \in \mathfrak{h}$. The Verma module M_λ over $U_q(\mathfrak{g})$ is generated by v_λ with defining relations $e_i v_\lambda = 0$, $q^h v_\lambda = q^{\lambda(h)} v_\lambda$.

Remark. All the Verma modules in this paper are equipped with a distinguished generator (i.e. the normalization of the generator is fixed).

We say that an object V in $\mathcal{O}$ is integrable if for all i, it is a sum of finite dimensional submodules with respect to the subalgebra generated by e_i, f_i, q^{bh_i}, $b \in \mathbb{C}$.

We say that λ is a dominant integral weight if $\lambda(h_i)$ is a nonnegative integer for all i. The set of dominant integral weights is denoted by P_+.

The irreducible module L_λ with highest weight λ over $U_q(\mathfrak{g})$ is an integrable module if and only if λ is a dominant integral weight. The category $\mathcal{O}_{int}$ of integrable modules is semisimple, with irreducible objects being L_λ for dominant integral λ. This category is closed under tensor product.

2.4. **The Weyl group.** Recall that the Weyl group $\mathbb{W}$ of $\mathfrak{g}$ is the group of transformations of $\mathfrak{h}$ generated by the reflections $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. It is known that the defining relations for $\mathbb{W}$ are: $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$, $i \neq j$, where $m_{ij} = 2, 3, 4, 6, \infty$ if $a_{ij}a_{ji} = 0, 1, 2, 3, \geq 4$ (if $m_{ij} = \infty$ then we agree that there is no relation). Any element of $\mathbb{W}$ is a product of s_i. The smallest number of factors in such a product is called the length of w and denoted by $l(w)$. A representation of $w \in \mathbb{W}$ by a product of length $l(w)$ is called a reduced decomposition.

The group $\mathbb{W}$ is the quotient of the group $\tilde{\mathbb{W}}$ generated by s_i with the braid relations

$$s_i s_j s_i ... = s_j s_i s_j ...$$

(m_{ij} terms on both sides), by the additional relations $s_i^2 = 1$. The group $\tilde{\mathbb{W}}$ is called the braid group of $\mathfrak{g}$.

Any two reduced decompositions of an element of $\mathbb{W}$ coincide not only in $\mathbb{W}$ but also in $\tilde{\mathbb{W}}$ (see e.g. [Lu], 2.1.2). Therefore, the projection map $\tilde{\mathbb{W}} \to \mathbb{W}$ admits a splitting

$\gamma : \mathbb{W} \to \tilde{\mathbb{W}}$, assigning to any element of $\mathbb{W}$ given by some reduced decomposition, the element of $\tilde{\mathbb{W}}$ defined by the same decomposition (of course, this is only a map of sets, not a group homomorphism). Using the map γ, we will regard $\mathbb{W}$ as a subset of $\tilde{\mathbb{W}}$.

Let us fix a weight ρ such that $\rho(h_i) = 1$ for all i. Define the shifted action of the Weyl group on $\mathfrak{h}$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$. It is obvious that the shifted action is independent on the choice of ρ.

2.5. **Intertwining operators and expectation values.** Let V be an $\mathfrak{h}$-diagonalizable module over $U_q(\mathfrak{g})$, and $\Phi : M_\lambda \to M_\mu \otimes V$ an intertwining operator. We have $\Phi v_\lambda = v_\mu \otimes v + ...$, where ... denote terms of lower weight in the first component, and $v \in V[\lambda - \mu]$. We will call v the expectation value of Φ and write $< \Phi >= v$.

Let V be a module over $U_q(\mathfrak{g})$ which belongs to $\mathcal{O}$. Let ν be a weight of V.

Lemma 1. *For generic λ the map $\Phi \to < \Phi >$ is an isomorphism of vector spaces $Hom_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\nu} \otimes V) \to V[\nu]$. In particular, this map is an isomorphism for dominant weights with sufficiently large coordinates $\lambda(h_i)$ for all $i = 1, ..., r$.*

Proof. The proof is straightforward; see [ES] and [ESt]. □

This lemma allows one to define, for $v \in V[\mu]$ and generic $\lambda \in \mathfrak{h}^*$, the intertwining operator Φ^v_λ such that $< \Phi^v_\lambda >= v$. It is easy to see that the matrix elements of this operator with respect to the bases in M_λ, $M_{\lambda-\mu}$ induced by any bases in $U_q(\mathfrak{n}_-)$ and V, are rational functions of (λ, α_i) for $q = 1$ and of $q^{(\lambda,\alpha_i)}$ if $q \neq 1$.

2.6. **Fusion and exchange matrices.** Recall the definition of the fusion and exchange matrices [ES, EV1].

Let $\lambda \in \mathfrak{h}^*$ be a generic weight. Let V, U be integrable $U_q(\mathfrak{g})$-modules, and $v \in V[\mu]$, $u \in U[\nu]$. Consider the composition

$$\Phi^{u,v}_\lambda : \; M_\lambda \xrightarrow{\Phi^v_\lambda \otimes 1} M_{\lambda-\mu} \otimes V \xrightarrow{\Phi^u_{\lambda-\mu} \otimes 1} M_{\lambda-\mu-\nu} \otimes U \otimes V.$$

Then $\Phi^{u,v}_\lambda \in \mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\mu-\nu} \otimes U \otimes V)$. We will call $\Phi^{u,v}_\lambda$ the *fusion* of Φ^v_λ and $\Phi^u_{\lambda-\mu}$.

For a generic λ there exists a unique element $x \in (U \otimes V)[\mu + \nu]$ such that $\Phi^x_\lambda = \Phi^{u,v}_\lambda$. The assignment $(u, v) \mapsto x$ is bilinear, and defines a zero weight map

$$J_{UV}(\lambda) : \; U \otimes V \to U \otimes V.$$

The operator $J_{UV}(\lambda)$ is called the fusion matrix of U and V. The fusion matrix $J_{UV}(\lambda)$ is a rational function of λ for $q = 1$ (respectively of q^λ if $q \neq 1$).

Also, $J_{UV}(\lambda)$ is strictly lower triangular, i.e. $J = 1 + L$ where $L(U[\nu] \otimes V[\mu]) \subset \oplus_{\tau<\nu,\, \mu<\sigma} U[\tau] \otimes V[\sigma]$. In particular, $J_{UV}(\lambda)$ is invertible.

The exchange matrix $R_{VU}(\lambda)$ is defined by the formula

$$R_{VU}(\lambda) = J_{VU}(\lambda)^{-1} \mathcal{R}^{21} J^{21}_{UV}(\lambda),$$

where $\mathcal{R}$ is the universal R-matrix of $U_q(\mathfrak{g})$. The exchange matrix has zero weight. It is also called a dynamical R-matrix, since it satisfies the quantum dynamical Yang-Baxter equation (see [EV1]).

In the theory of fusion and exchange matrices, one often uses the so called "dynamical notation", which will also be useful for us here. This notation is defined as follows.

Let $V_1, \dots V_n$ be $\mathfrak{h}^*$-graded vector spaces, and let $F(\lambda) : V_1 \otimes \dots \otimes V_n \to V_1 \otimes \dots \otimes V_n$ be a linear operator depending on $\lambda \in \mathfrak{h}^*$. Then for any homogeneous $u_1, \dots, u_n$, $u_i \in V_i[\nu_i]$, we define $F(\lambda - h^{(i)})(u_1 \otimes \dots \otimes u_n)$ to be $F(\lambda - \nu_i)(u_1 \otimes \dots \otimes u_n)$. In particular, when $n = 1$, we will denote the term $h^{(1)}$ simply by h: that is, $F(\lambda - h)v = F(\lambda - \nu)v$ if v has weight ν.

2.7. **Singular vectors in Verma modules.** Recall that a nonzero vector in a $U_q(\mathfrak{g})$-module is said to be singular if it is annihilated by e_i for all i.

Let $w \in \mathbb{W}$ and $w = s_{i_1} \dots s_{i_l}$ be a reduced decomposition. Set $\alpha^l = \alpha_{i_l}$ and $\alpha^j = (s_{i_l} \dots s_{i_{j+1}})(\alpha_{i_j})$ for $j = 1, \dots, l-1$. Let $n_j = 2\frac{(\lambda+\rho,\alpha^j)}{(\alpha^j,\alpha^j)}$. For a dominant $\lambda \in P_+$, n_j are positive integers. Let $d^j = d_{i_j}$ (where d_i are the symmetrizing numbers).

We will need the following lemma, which is similar to Lemma 4 in [TV].

Lemma 2. *Let λ be a dominant integral weight. Then the collection of pairs of integers $(n_1, d^1), ..., (n_k, d^k)$ and the product $f_{\alpha_{i_1}}^{n_1} \cdots f_{\alpha_{i_l}}^{n_l}$ does not depend on the reduced decomposition.*

Remark. The second statement of the lemma is known as the "quantum Verma identities" and is proved in [Lu], Section 39.3. In the case $q = 1$, it goes back to Verma. However, we will give the argument (which is somewhat different from the one in [Lu]) for the reader's convenience.

Proof. It is sufficient to prove the statement for two reduced decompositions that can be identified by applying a braid relation once. Therefore, it is sufficient to check the statement for rank 2 Lie algebras A_2, B_2, G_2. In this case the only element that has two different reduced decompositions is the maximal element w_0. So we can assume that we are dealing with the two different reduced decompositions of $w = w_0$, namely, $w = s_{i_1} ... s_{i_l} = s_{i'_1} ... s_{i'_l}$.

Since $w = w_0$, it is easy to see that for either reduced decomposition, $\alpha^1, \dots, \alpha^l$ are all the positive roots (each occuring exactly once).

Hence, the collection $(n_1, d^1) \dots (n_l, d^l)$ does not depend on the reduced decomposition.

Let n_i, n'_i be the numbers defined above for the two decompositions. The vectors $u = f_{\alpha_{i_1}}^{n_1} \cdots f_{\alpha_{i_l}}^{n_l} v_\lambda$, $u' = f_{\alpha_{i'_1}}^{n'_1} \cdots f_{\alpha_{i'_l}}^{n'_l} v_\lambda$ are singular vectors in M_λ of weight $w_0 \cdot \lambda$ (they are nonzero, since the algebra $U_q(\mathfrak{n}_-)$ has no zero divisors).

For Lie algebras A_2, B_2, G_2, it is easy to see that the space of singular vectors in M_λ in the weight $w_0 \cdot \lambda$ is 1-dimensional. Indeed, the module $M_{w_0 \cdot \lambda}$ is irreducible, so if there were two independent singular vectors of weight $w_0 \cdot \lambda$, then the direct sum of two copies of $M_{w_0 \cdot \lambda}$ would be contained in M_λ. But this is impossible, since some weight

multiplicities of this direct sum are bigger than the corresponding weight multiplicities of M_λ.

Therefore, the vectors u, u' are proportional. Since M_λ is a free module over the subalgebra $U_q(\mathfrak{n}_-)$, we have $f^{n'_1}_{\alpha_{i'_1}} \dots f^{n'_l}_{\alpha_{i'_l}} = c\, f^{n_1}_{\alpha_{i_1}} \dots f^{n_l}_{\alpha_{i_l}}$ in $U_q(\mathfrak{n}_-)$ for a suitable $c \in \mathbb{C}^*$.

We claim that $c = 1$. Indeed, consider the natural homomorphism from $U_q(\mathfrak{n}_-)$ to the algebra generated by f_i with the relations $f_i f_j = q_i^{a_{ij}} f_j f_i$, $i < j$, sending f_i to f_i (it is easy to check that such a homomorphism exists). The images of the two monomials under this homomorphism differ by a power of q, which implies that $c = q^m$. On the other hand, since the Serre relations are symmetric under $q \to q^{-1}$, a similar homomorphism exists if q is replaced with q^{-1}, which yields $c = q^{-m}$. Thus, $c = 1$, as desired. □

Let δ be a reduced decomposition of $w \in \mathbb{W}$ given by the formula $w = s_{i_1} ... s_{i_l}$. Define a vector $v^\lambda_{w\cdot\lambda,\delta} \in M_\lambda$ by

$$v^\lambda_{w\cdot\lambda,\delta} = \frac{f^{n_1}_{\alpha_{i_1}}}{[n_1]_{q^{d^1}}!} \cdots \frac{f^{n_l}_{\alpha_{i_l}}}{[n_l]_{q^{d^l}}!}\, v_\lambda\,, \tag{1}$$

This vector is singular. It does not depend on the reduced decomposition δ by Lemma 2, so we will often denote it by $v^\lambda_{w\cdot\lambda}$.

3. The main construction for $U_q(\mathfrak{sl}_2)$

3.1. **The operators** $A_{s,V}(\lambda)$**.** Let $\mathfrak{g} = \mathfrak{sl}_2$. Identify the space of weights for $\mathfrak{sl}_2$ with $\mathbb{C}$ by $z \in \mathbb{C} \to z\alpha/2$, where α is the positive root; then dominant integral weights are identified with positive integers.

Let s be the nontrivial element of the Weyl group of $\mathfrak{sl}_2$. Let V be a finite dimensional $U_q(\mathfrak{sl}_2)$-module, and let λ be a sufficiently large positive integer (compared to V). Define a linear operator $A_{s,V}(\lambda) : V \to V$ as follows.

Fix a weight ν of V. Let $v \in V[\nu]$. Consider the intertwining operator $\Phi^v_\lambda : M_\lambda \to M_{\lambda-\nu} \otimes V$. By Lemma 1, such an operator is well defined.

Lemma 3. *For a sufficiently large positive integer λ, there exists a unique linear operator $A_{s,V}(\lambda) : V \to V$ such that*

$$\Phi^v_\lambda v^\lambda_{s\cdot\lambda} = v^{\lambda-\nu}_{s\cdot(\lambda-\nu)} \otimes A_{s,V}(\lambda)v + \textit{lower weight terms}.$$

(where the weight is taken in the first component). This operator is invertible.

Proof. The proof of existence and uniqueness of $A_{s,V}$ is straightforward (see e.g. [TV]). To prove the invertibility, it is sufficient to observe that for an irreducible module V and large λ, the map $A_{s,V}(\lambda) : V[\nu] \to V[-\nu]$ is nonzero. Indeed, the tensor product of a Verma module with a finite dimensional module does not contain finite dimensional submodules. Thus, the operator Φ^v_λ cannot have finite rank, and hence has to be nonzero on $M_{-\lambda-2}$. □

Thus, the operator $A_{w,V}(\lambda)$ is the effect, at the level of expectation values, of the operation of **restriction** of an intertwiner from M_λ to $M_{s\cdot\lambda}$.

The goal of the next few sections is to compute the operator $A_{s,V}(\lambda)$. This can be done analogously to [TV], using a direct calculation and identities with hypergeometric functions. However, we would like to give a different derivation, which seems to be a bit simpler (in the spirit of [EV2], subsection 7.2).

The main tool of the calculation is the following important property of $A_{s,V}(\lambda)$.

Lemma 4. *Let U, V be finite dimensional $U_q(\mathfrak{sl}_2)$-modules. Then*

$$A_{s,U\otimes V}(\lambda)J_{UV}(\lambda) = J_{UV}(s\cdot\lambda)A^{(2)}_{s,V}(\lambda)A^{(1)}_{s,U}(\lambda - h^{(2)}) \tag{2}$$

where $A^{(1)}$ denotes $A\otimes 1$, $A^{(2)}$ denotes $1\otimes A$.

Proof. The lemma is an easy consequence of the definitions: it expresses the fact that the operation of fusion of intertwiners commutes with the operation of restriction of intertwiners to submodules. □

3.2. **Calculation of $A_{s,V}(\lambda)$ in the 2-dimensional representation.** For brevity we denote $A_{s,V}$ by A_V. Consider the case when V is the 2-dimensional irreducible representation with the standard basis v_+ and v_-, such that $ev_+ = fv_- = 0$, $ev_- = v_+$, $fv_+ = v_-$, $q^{bh}v_\pm = q^{\pm b}v_\pm$.

Lemma 5. *One has*

$$A_V(\lambda)v_+ = qv_-, \quad A_V(\lambda)v_- = -q^{-1}\frac{[\lambda+2]_q}{[\lambda+1]_q}v_+.$$

Proof. Consider the intertwiner $\Phi^{v_+}_\lambda$. It satisfies the relation

$$\Phi^{v_+}_\lambda v_\lambda = v_{\lambda-1}\otimes v_+.$$

Therefore,

$$\Phi^{v_+}_\lambda \frac{f^{\lambda+1}}{[\lambda+1]_q!}v_\lambda = \frac{1}{[\lambda+1]_q!}(f\otimes 1 + q^{-h}\otimes f)^{\lambda+1}(v_{\lambda-1}\otimes v_+)$$

This implies after a straightforward calculation that

$$A_V(\lambda)v_+ = qv_-.$$

Now let us consider the intertwiner $\Phi^{v_-}_\lambda$. It satisfies the relation

$$\Phi^{v_-}_\lambda v_\lambda = v_{\lambda+1}\otimes v_- - q^{-1}[\lambda+1]_q^{-1}fv_{\lambda+1}\otimes v_+.$$

Therefore,

$$\Phi^{v_-}_\lambda \frac{f^{\lambda+1}}{[\lambda+1]_q!}v_\lambda = \frac{1}{[\lambda+1]_q!}(f\otimes 1 + q^{-h}\otimes f)^{\lambda+1}(v_{\lambda+1}\otimes v_- - q^{-1}[\lambda+1]_q^{-1}fv_{\lambda+1}\otimes v_+).$$

This implies that

$$A_V(\lambda)v_- = -q^{-1}\frac{[\lambda+2]_q}{[\lambda+1]_q}v_+,$$

as desired. $\square$

3.3. **The calculation in any finite dimensional representation (up to a constant).** Now we let $V = V_m$ be the irreducible representation with highest weight m. For any $k = 0, ..., m$, define a linear map $A_m^k(\lambda) : V[m-2k] \to V[2k-m]$ to be the restriction of $A_V(\lambda)$ to $V[m-2k]$. If we choose generators of the 1-dimensional spaces $V[m-2k]$, this map will be expressed by a scalar complex valued function of λ.

Up to a λ-independent factor, this function is independent on the choice of the generators. Thus, we can naturally understand $A_m^k(\lambda)$ as an element of the group

$$(\text{nonvanishing complex valued functions on } \mathbb{Z}_+ + N)/\mathbb{C}^*.$$

(where N is a large enough number). This will be our point of view in this subsection. The equality of two elements in this group will be denoted by the sign $\equiv$.

Proposition 6. *One has*

$$A_m^k(\lambda) \equiv \prod_{j=1}^{k} \frac{[\lambda+1+j]_q}{[\lambda-m+k+j]_q}.$$

Remark. In the case $q = 1$, this proposition appears in [TV].

Proof. Let $m \geq 1$. Consider equation (2) in the weight subspace of weight $m-2k+1$ in the tensor product $V_1 \otimes V_m$. Let us identify this weight subspace with the opposite one in any way, and take the determinant of both sides of (2).

Since the fusion matrix is triangular with the diagonal elements equal to 1, its determinant is 1. Therefore, using the decomposition $V_1 \otimes V_m = V_{m-1} \oplus V_{m+1}$ we obtain for $k = 0$:

$$A_{m+1}^0(\lambda) \equiv A_m^0(\lambda) A_1^0(\lambda - m),$$

and for $k \neq 0$

$$A_{m+1}^k(\lambda) A_{m-1}^{k-1}(\lambda) \equiv A_m^k(\lambda) A_m^{k-1}(\lambda) A_1^0(\lambda - m + 2k) A_1^1(\lambda - m + 2k - 2).$$

Now let us substitute the values of A_1^0 and A_1^1 computed in the previous section. Then we get

$$A_m^0(\lambda) = 1,$$

$$A_{m+1}^k(\lambda) \equiv \frac{[\lambda - m + 2k]_q}{[\lambda - m + 2k - 1]_q} A_m^k(\lambda) A_m^{k-1}(\lambda) A_{m-1}^{k-1}(\lambda)^{-1}$$

It is clear that A_m^k is completely determined from this equation. It remains to check that the expression given in the proposition satisfies the equations, which is straightforward. $\square$

Corollary 7. *The operator-valued function $A_V(\lambda)$, defined for large positive integers, uniquely extends to a rational function of λ (for $q = 1$) and of q^λ (for $q \neq 1$). For generic λ, the operator $A_{s,V}(\lambda)$ is invertible.*

Proof. The existence follows from the above explicit computation of A_V. The uniqueness is obvious, since the function is defined at infinitely many points. The invertibility follows from Lemma 3. □

3.4. **Limits of $A_V(\lambda)$ at infinity.** In the previous subsection, we have calculated A_V up to a constant. In this subsection, we will explicitly calculate this constant.

Let V be a finite dimensional representation of $U_q(\mathfrak{sl}_2)$. Proposition 6 implies the following result.

Corollary 8. *(i) If $q = 1$ then the map $A_V(\lambda)$ has a limit $A_V^{\pm} = A_V^{\infty}$ at $\lambda = \pm\infty$. If $q \neq 1$, the map $A_V(\lambda)$ has a limit A_V^+ as $q^\lambda \to \infty$ and A_V^- as $q^\lambda \to 0$, respectively.*

(ii) One has

$$A_V^-|_{V_m[m-2k]} = q^{-2k(m-k+1)} A_V^+|_{V_m[m-2k]}.$$

In other words, one has

$$A_V^- = A_V^+ \mathbf{u} q^{-h^2/2},$$

where $\mathbf{u}$ is the Drinfeld element $m_{21}(S \otimes 1)\mathcal{R}$, [Dr] *(here $m_{21}(a \otimes b) = ba$).*

(iii) Define

$$A_V^{\infty}|_{V_m[m-2k]} = q^{-k(m-k+1)} A_V^+|_{V_m[m-2k]} = q^{k(m-k+1)} A_V^-|_{V_m[m-2k]}.$$

to be the geometric mean of the two limits. Then one has

$$A_V(\lambda) = A_V^{\infty} B_V(\lambda),$$

where $B_V(\lambda)$ is a weight zero operator, defined by

$$B_{V_m}(\lambda)|_{V_m[m-2k]} = \prod_{j=1}^{k} \frac{[\lambda + 1 + j]_q}{[\lambda + 1 - m + 2k - j]_q}.$$

Remark. Recall that the element $\mathbf{u}$ acts on V_m as $q^{-m(m+2)/2} q^h$.

3.5. **Computation of $A_V^{\pm}, A_V^{\infty}$.** Let $\mathcal{R}_0 = \mathcal{R} q^{-h \otimes h/2}$.

Proposition 9. *One has*

$$A_{U \otimes V}^+ = \mathcal{R}_0^{21}(A_U^+ \otimes A_V^+).$$

Proof. Theorem 50 from [EV1] implies that as $q^\lambda \to \infty$, one has $J_{UV}(\lambda) \to 1$ and $J_{UV}(-\lambda) \to \mathcal{R}_0^{21}$. Thus, going to the limit $q^\lambda \to \infty$ in (2), and using Theorem 50 from [EV1], we obtain the proposition. □

Remark 1. We take this opportunity to correct the statement of Theorem 50 of [EV1]. First of all, the condition $|q| < 1$ should be replaced with $|q| > 1$. (The paper [ESt], whose results are used to prove the theorem, refers to the comultiplication opposite to that of [EV1], and the two comultiplications are related by the transformation $q \to q^{-1}$; cf. formula (45) in [EV1]). Second, $\mathfrak{n}_{\pm}$ should be replaced by $\mathfrak{b}_{\pm}$ in the line preceding the theorem.

Remark 2. Another proof of Theorem 50 of [EV1] (different from the original one) can be obtained by sending λ to ∞ in the ABRR equation (see [ABRR, ES]).

Define $A'_V := A^+_V q^{\frac{h(h+2)}{4}}$. It follows from Corollary 8, part (iii) that on V_m, one has $A'_V = A^\infty_V q^{m(m+2)/4}$.

Corollary 10. *One has*

$$A'_{V\otimes U} = \mathcal{R}^{21}(A'_V \otimes A'_U).$$

Proof. Straightforward from Proposition 9. ☐

Proposition 11. *One has in V:*

$$A'_V f = -q^{-2} e A'_V,\ A'_V e = -q^2 f A'_V.$$

and

$$A^\infty_V f = -q^{-2} e A^\infty_V,\ A^\infty_V e = -q^2 f A^\infty_V.$$

Proof. We prove the relations for A'_V; the relations for A^∞_V follow automatically since these two operators are proportional.

If V is 2-dimensional, then A'_V is known, and it is straightforward to establish the result. From this and Corollary 10 it follows that the result is true if V is the tensor product of any number of 2-dimensional representations. But any finite dimensional representation is contained in such a product, so we are done. ☐

Now let v_m be a highest weight vector of $V = V_m$. Let us compute the operator of A'_V in the basis $v_{m-2j} := \frac{f^j}{[j]_q!} v_m$, $j = 0, ..., m$.

Proposition 12. *One has*

$$A^\infty_V v_{m-2j} = (-1)^j q^{m-2j} v_{2j-m}.$$

Proof. First of all observe that in $V_1^{\otimes m}$, one has $\Delta_m(f^m) v_+^{\otimes m} = [m]_q! v_-^{\otimes m}$ (where Δ_m is the iterated coproduct). On the other hand, using the expression for A^+_V for the 2-dimensional V, and the expression for $A^+_{V_1\otimes V_2}$ given in Proposition 9, we get

$$A^+_{V_1^{\otimes m}}(v_+^{\otimes m}) = q^m (v_-^{\otimes m}).$$

But V_m is the submodule of $V_1^{\otimes m}$ generated by $v_+^{\otimes m}$. Thus, for

$$A^\infty_V v_m = A^+_V v_m = q^m v_{-m}.$$

Now the result follows from Proposition 11 by a direct calculation. ☐

3.6. **The operators $B_V^{\pm}(\lambda)$.** Following [TV], define the operators $B_V^{\pm}(\lambda)$ by the formula

$$B_V^{\pm}(\lambda) = (A_V^{\pm})^{-1} A_V(\lambda).$$

Proposition 13. *(i) $B_V^{\pm}$ preserve the weight decomposition.*

(ii) $B_V^{+} \to 1$ as $q^{\lambda} \to +\infty$; $B_V^{-} \to 1$ as $q^{\lambda} \to 0$.

(iii) One has

$$B_{V_m}^{\pm}(\lambda)|_{V_m[m-2k]} = \prod_{j=1}^{k} \frac{(\lambda+1+j)_{q^{\mp 2}}}{(\lambda+1-m+2k-j)_{q^{\mp 2}}},$$

where $(a)_q := \frac{q^a-1}{q-1}$ is the nonsymmetric q-analog of a.

(iv) If $q=1$, then B_V^{+}, B_V^{-} are equal to the operator B_V from Corollary 8.

Proof. The proof of this proposition is straightforward from the previous results. □

Proposition 14. *The operator $B_V^{+}(\lambda)$ is given by the action in V of the universal element*

$$B^{+}(\lambda) = p(\lambda, h, e, f),$$

where

$$p(\lambda, h, e, f) := \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{q^{-k(\lambda+1)}}{[k]_q!} f^k e^k \prod_{\nu=0}^{k-1} \frac{1}{[\lambda - h - \nu]_q}.$$

In particular, for $q=1$, the operator $B^{\pm} = B$ coincides with the operator B from section 2.5 of [TV].

Remark. Similar formulas can be deduced for B^{-} and B.

Proof. This is proved by a straightforward calculation with intertwiners, which generalizes to the q-case the calculation of [TV]. Another proof is given as a remark in Section 6. □

4. The main construction for any $\mathfrak{g}$: the dynamical Weyl group

4.1. **The operators $A_{w,V}(\lambda)$.** Let us return to the situation of a general $\mathfrak{g}$. Let V be an integrable $U_q(\mathfrak{g})$-module, and ν be a weight of V. For any simple reflection $s_i \in \mathbb{W}$, we define an operator-valued rational function $A_{s_i,V}(\lambda) : V[\nu] \to V[s_i\nu]$ by the formula

$$A_{s_i,V}(\lambda) = A_{s,V'}(\lambda(h_i)),$$

where V' is the $U_{q_i}(\mathfrak{sl}_2)$ submodule of V generated by $V[\nu]$.

Let $w \in \mathbb{W}$, and let $w = s_{i_1}...s_{i_l}$ be a reduced decomposition. Let us call this reduced decomposition δ.

Definition: Define the operator $A_{w,V,\delta}(\lambda) : V[\nu] \to V[w\nu]$ by

$$A_{w,V,\delta}(\lambda) = A_{s_{i_1}}(s_{i_2}...s_{i_l} \cdot \lambda)...A_{s_{i_{l-1}}}(s_{i_l} \cdot \lambda) A_{s_{i_l}}(\lambda).$$

Thus, the function $A_{w,V,\delta}(\lambda)|_{V[\nu]}$ uniquely extends to a rational operator valued function of the variables (λ, α_i), respectively $q^{(\lambda,\alpha_i)}$. This function is generically invertible.

The following two results play a crucial role in our considerations.

Let $\Phi : M_\lambda \to M_\mu \otimes V$ be an intertwiner, $\Phi v_\lambda = v_\mu \otimes < \Phi > + ...$ Assume that λ is dominant, and $\lambda(h_i)$ are sufficiently large for all i.

Proposition 15. *For any δ and $w \in \mathbb{W}$, one has*

$$\Phi v^\lambda_{w\cdot\lambda,\delta} = v^\mu_{w\cdot\mu,\delta} \otimes A_{w,V,\delta}(\lambda) < \Phi > + \text{lower weight terms.}$$

Proof. The statement follows easily from the definitions by induction on the length of w. □

Corollary 16. *The operator $A_{w,V,\delta}(\lambda)$ is independent of the reduced decomposition δ of w.*

Proof. If λ is large dominant, the statement is clear from Proposition 15, since $v^\lambda_{w\cdot\lambda,\delta}$ is independent of the reduced decomposition δ of w by the quantum Verma identities (Lemma 2). For an arbitrary λ, the statement follows from the fact that a rational function is completely determined by its values at large dominant weights. □

Thus, we will denote $A_{w,V,\delta}(\lambda)$ simply by $A_{w,V}(\lambda)$. Proposition 15 shows that as for $U_q(\mathfrak{s}l_2)$, the operator $A_{w,V,\delta}(\lambda)$ is the effect, at the level of expectation values, of the restriction of an intertwiner from M_λ to $M_{w\cdot\lambda}$.

The following lemma is an easy consequence of the definition.

Lemma 17. *If $w_1, w_2 \in \mathbb{W}$, $l(w_1w_2) = l(w_1) + l(w_2)$, then*

$$A_{w_1w_2,V}(\lambda) \;=\; A_{w_1,V}(w_2\cdot\lambda)A_{w_2,V}(\lambda)$$

on $V[\nu]$.

More generally, we can define the operators $A_{w,V}(\lambda)$ for any element w of the braid group $\tilde{\mathbb{W}}$. Namely, let s_i^{-1} be the inverse of s_i in $\tilde{\mathbb{W}}$, and define the operator $A_{s_i^{-1},V}(\lambda)$ by

$$A_{s_i^{-1},V}(\lambda) = A_{s_i,V}(s_i\cdot\lambda)^{-1}.$$

Now, if $w = s_{i_1}^{\varepsilon_1}...s_{i_l}^{\varepsilon_l}$ where $\varepsilon_i = \pm 1$, then we define $A_{w,V}(\lambda)$ by the formula

$$A_{w,V}(\lambda) = A_{s_{i_1}^{\varepsilon_1}}(s_{i_2}...s_{i_l}\cdot\lambda)...A_{s_{i_{l-1}}^{\varepsilon_{l-1}}}(s_{i_l}\cdot\lambda)A_{s_{i_l}^{\varepsilon_l}}(\lambda).$$

It is easy to see from Corollary 16 or Lemma 17 that this definition is independent on the product representation of w but depends only of w itself, so the notation $A_{w,V}$ is non-ambiguous.

4.2. **The dynamical Weyl group.** The above results imply that using the operators $A_{w,V}(\lambda)$, one can define a $\mathbb{C}$-linear action of the braid group $\tilde{\mathbb{W}}$ on the space of meromorphic functions of λ with values in V, by the formula

$$(w \circ f)(\lambda) = A_{w,V}(w^{-1} \cdot \lambda) f(w^{-1} \cdot \lambda), w \in \tilde{\mathbb{W}}$$

Now we will give the main definitions of this paper.

Main definition 1: Let V be an integrable $U_q(\mathfrak{g})$-module. The $\tilde{\mathbb{W}}$-action $f \to w \circ f$ on V-valued functions on $\mathfrak{h}^*$ will be called **the shifted dynamical action**.

We would also like to give a name to the operators $A_{w,V}(\lambda)$, since they play a central role in the paper. We defined these operators for all $w \in \tilde{\mathbb{W}}$. However, the operators $A_{w,V}$ corresponding to elements w of $\mathbb{W}$ (regarded as a subset of $\tilde{\mathbb{W}}$) are especially remarkable and occur especially often in applications. Therefore, we will restrict our attention to them, and make the following definition.

Main definition 2: We call the collection of operator valued rational functions $A_{w,V}(\lambda)$, $w \in \mathbb{W}$ **the dynamical Weyl group** of V.

Remark. One of the important properties of dynamical R-matrices is that they tend to usual R-matrices when the dynamical parameter λ goes to infinity (see [EV1], Theorem 50). On the other hand, we will show later that when λ goes to infinity, the operators $A_{w,V}(\lambda)$ tend to elements of the usual classical or quantum Weyl group acting in V. This justifies the term "dynamical Weyl group".

We also define the (unshifted) **dynamical action** $w \to w*$ of $\tilde{\mathbb{W}}$ on functions of λ with values in V as follows. We introduce the operators

$$\mathcal{A}_{w,V}(\lambda) = A_{w,V}(-\lambda - \rho + \frac{1}{2}h).$$

Then by the definition

$$(w * f)(\lambda) = \mathcal{A}_{w,V}(w^{-1}\lambda) f(w^{-1}\lambda).$$

Remark. Observe that for $w_1, w_2 \in \mathbb{W}$, such that $l(w_1 w_2) = l(w_1) + l(w_2)$, one has

$$\mathcal{A}_{w_1 w_2,V}(\lambda) = \mathcal{A}_{w_1,V}(w_2\lambda)\mathcal{A}_{w_2,V}(\lambda).$$

This implies that $w*$ is indeed an action of $\tilde{\mathbb{W}}$.

4.3. **The dynamical Weyl group of the tensor product and the dual representation.** The following is one of the main properties of the dynamical Weyl group.

Lemma 18. *Let U, V be integrable $U_q(\mathfrak{g})$-modules. Let $w \in \mathbb{W}$. Then*

$$A_{w,U\otimes V}(\lambda) J_{UV}(\lambda) \;=\; J_{UV}(w \cdot \lambda) A^{(2)}_{w,V}(\lambda) A^{(1)}_{w,U}(\lambda - h^{(2)}) \tag{3}$$

where $A^{(1)}$ denotes $A \otimes 1$, and $A^{(2)}$ denotes $1 \otimes A$.

Proof. As for $U_q(\mathfrak{sl}_2)$, the lemma is an easy consequence of the definitions: it expresses the fact that the operation of fusion of intertwiners commutes with the operation of restriction of intertwiners to submodules. ∎

Corollary 19. *For any $w \in \mathbb{W}$ the dynamical R-matrix $R_{VU}(\lambda)$ satisfies the relation*

$$R_{VU}(w\cdot\lambda) = A^{(2)}_{w,U}(\lambda)A^{(1)}_{w,V}(\lambda - h^{(2)})R_{VU}(\lambda)A^{(2)}_{w,U}(\lambda - h^{(1)})^{-1}A^{(1)}_{w,V}(\lambda)^{-1}$$

Remark. Here $A^{(2)}_{w,U}(\lambda - h^{(1)})^{-1}$ is the inverse of the operator $A^{(2)}_{w,U}(\lambda - h^{(1)})$.

Now let $\mathfrak{g}$ be finite dimensional, and let us calculate the dynamical Weyl group operators on the dual to a given representation.

Recall that the dual space U^* of a $U_q(\mathfrak{g})$-module U has two module structures. The first one is given by $a \to (S(a)|_U)^*$ and the second one is given by $a \to (S^{-1}(a)|_U)^*$. The corresponding $U_q(\mathfrak{g})$-modules are denoted U^* and ${}^*U$. These modules are isomorphic, via $q^{2\rho} : {}^*U \to U^*$.

Let $Q_U(\lambda) : U \to U$ be given by $Q_U(\lambda) = \sum (c_i')^* c_i$, where $\sum c_i \otimes c_i' = J_{U,{}^*U}(\lambda)$ (this operator was introduced in [EV1, EV2]).

Proposition 20. *For any $w \in \mathbb{W}$ one has*

$$A_{w,{}^*U}(\lambda)^* = Q_U(\lambda)A_{w,U}(\lambda - h)^{-1}Q_U(w\cdot\lambda)^{-1},$$

and hence

$$A_{w,U^*}(\lambda)^* = q^{2\rho}Q_U(\lambda)A_{w,U}(\lambda - h)^{-1}Q_U(w\cdot\lambda)^{-1}q^{-2\rho}.$$

Remark. This formula has recently been used in the theory of trace functionals on non-commutative moduli spaces of flat connections, see [RS].

Proof. It is enough to establish the first formula. The proof of this formula is obtained by specializing formula (3) to the case $V = {}^*U$, dualization of the second component, and multiplication of the components. □

Here is another formula for the dynamical Weyl group of the dual representation, which is valid on the zero weight subspace, and involves a sign change for λ.

Proposition 21. *One has*

$$\mathcal{A}_{w,U^*}(w^{-1}\lambda)^*|_{U[0]} = \mathcal{A}_{w,{}^*U}(w^{-1}\lambda)^*|_{U[0]} = \mathcal{A}_{w^{-1}}(-\lambda)|_{U[0]}$$

Proof. Let $w = s_{i_1}...s_{i_l}$ be a reduced decomposition. Using the Main Definition, we have

$$\mathcal{A}_{w,U^*}(w^{-1}\lambda) = \mathcal{A}_{s_{i_1},U^*}(s_{i_2}...s_{i_l}w^{-1}\lambda)...\mathcal{A}_{s_{i_l},U^*}(w^{-1}\lambda) = \mathcal{A}_{s_{i_1},U^*}(s_{i_1}\lambda)...\mathcal{A}_{s_{i_l},U^*}(w^{-1}\lambda) =$$

$$\mathcal{A}_{s_{i_1},U^*}(-\lambda)...\mathcal{A}_{s_{i_l},U^*}(-s_{i_{l-1}}...s_{i_1}\lambda)$$

(in the last equality we used that $A_{s_i,V}(\lambda)$ depends only of (λ,α_i)). This implies that

$$\mathcal{A}_{w,U^*}(w^{-1}\lambda)^* = \mathcal{A}_{s_{i_l},U^*}(-s_{i_{l-1}}...s_{i_1}\lambda)^*...\mathcal{A}_{s_{i_1},U^*}(-\lambda)^*$$

But for $\mathfrak{g} = \mathfrak{s}l_2$ we obviously have

$$\mathcal{A}_{s,U^*}(\lambda)^* = \mathcal{A}_{s,U}(\lambda)$$

on $U[0]$. This, together with the Main Definition implies the result. □

5. Further properties of the dynamical Weyl group

5.1. **Limits of $A_{w,V}(\lambda)$ for any $\mathfrak{g}$.** Let $\mathfrak{g}$ be any Kac-Moody algebra, and $w \in \mathbb{W}$. Let V be an integrable $U_q(\mathfrak{g})$-module. For a root α, let $\varepsilon_\alpha(C)$ be the sign of (λ,α) in a Weyl chamber C.

Proposition 22. *(i) If $q=1$, then the map $A_{w,V}(\lambda)$ has a limit $A^\infty_{w,V}$ at $\lambda\to\infty$ in a generic direction. If $q\neq 1$, then for any Weyl chamber C the map $A_{w,V}(\lambda)$ has a limit $A^C_{w,V}$ as $\lambda\to\infty$ in a generic direction in the chamber C.*

(ii) Let us agree that for $q=1$, $A^C_{w,V}:=A^\infty_{w,V}$ for all C. Then, for every reduced decomposition $w=s_{i_1}...s_{i_l}$, and any q, one has

$$A^C_{w,V}=A^{\varepsilon_{\alpha^1}(C)}_{s_{i_1},V}...A^{\varepsilon_{\alpha^l}(C)}_{s_{i_l},V},$$

where $A^\pm_{s_i,V}$ are the elements $A^\pm_{s,V}$ for the subalgebras generated by e_i,f_i,h_i (for $q=1$) or e_i,f_i,q^{bh_i} (for $q\neq 1$).

Proof. The proof follows easily from the results of the previous section and the Main Definition. □

We will mostly consider the operators $A^C_{w,V}$ if C is the dominant or the antidominant chamber. In this case we will denote $A^C_{w,V}$ by $A^+_{w,V}$ and $A^-_{w,V}$, respectively.

Part (ii) of Proposition 22 implies the following statements

Corollary 23. *The assignments $s_i\to A^+_{s_i,V}$, $s_i\to A^-_{s_i,V}$ extend to actions of $\tilde{\mathbb{W}}$ on V.*

Proof. Clear. □

Corollary 24. *On $V[0]$, one has $A^-_{w,V}A^+_{w^{-1},V}=1$ for any $w\in\mathbb{W}$.*

Proof. To prove the statement for any Lie algebra, it is enough to do so for $\mathfrak{g}=\mathfrak{sl}_2$, in which case the result is an easy consequence of the results of Section 3. □

5.2. **The relation to the quantum Weyl group.** In this subsection we will discuss the relationship of the dynamical Weyl group with the quantum Weyl group for $U_q(\mathfrak{g})$.

The quantum Weyl group was introduced by Soibelman for finite dimensional $\mathfrak{g}$ (see [KoSo]) and by Lusztig in the Kac-Moody case (see [Lu]). It is discussed in many books and papers, which use various (though pairwise equivalent) conventions. We will use the conventions of the paper [Sa].

The quantum Weyl group element $\mathbb{S}$ in a completion of $U_q(\mathfrak{sl}_2)$ is defined by the formula [Sa]:

$$\mathbb{S}=\exp_{q^{-1}}(q^{-1}eq^{-h})\exp_{q^{-1}}(-f)\exp_{q^{-1}}(qeq^h)q^{h(h+1)/2},$$

where $\exp_p(x)=\sum_{m\geq 0}\frac{p^{m(m-1)/2}}{[m]_p!}x^m$. According to [Sa], Proposition 1.2.1, this element acts in $V=V_m$ as

$$\mathbb{S}v_{m-2j}=(-1)^{m-j}q^{(m-j)(j+1)}v_{2j-m}$$

Therefore, in V we have:

$$\mathbb{S}|_{V[m-2j]} = (-1)^m q^{(m-j+1)j} A_V^\infty = (-1)^m A_V^+.$$

Proposition 25. *For a finite dimensional $U_q(\mathfrak{sl}_2)$-module V, one has*

$$A_V^+ = (-1)^h \mathbb{S}, \quad A_V^- = q^h \mathbb{S}^{-1}$$

where $(-1)^h$ is the transformation acting by -1 on even-dimensional irreducible modules, and by 1 on odd dimensional ones (i.e. the quantum analog of the element -1 of the group $SL(2)$).

Proof. This follows at once by comparing the actions of both sides on basis vectors of V. □

Now let $\mathfrak{g}$ be a Kac-Moody algebra. Following [Sa], define the element S_i to be the element $\mathbb{S}$ of the simple root subalgebra of $U_q(\mathfrak{g})$ corresponding to the simple root α_i. The elements S_i define operators on any integrable module (the "symmetries of an integrable module" defined by Lusztig [Lu]).

As an immediate corollary of Proposition 25, we obtain the following well known and important result (see [Lu],[Sa]).

Corollary 26. *The elements S_i satisfy the braid relations of $\tilde{\mathbb{W}}$ in any integrable module.*

Proof. The result follows immediately from the equality $A_V^- = q^h \mathbb{S}^{-1}$ and Proposition 22. □

Remark. In particular, this result is valid for $q = 1$ and yields a braid group action on integrable modules over a classical Kac-Moody algebra. This action factorizes through an extension of $\mathbb{W}$ by a group isomorphic to $(\mathbb{Z}/2)^r$.

For $w \in s_{i_1}^{\varepsilon 1} ... s_{i_l}^{\varepsilon_l} \in \tilde{\mathbb{W}}$, let $\mathbb{S}_w$ be the quantum Weyl group operator corresponding to w (i.e. $\mathbb{S}_w = S_{i_1}^{\varepsilon_1} ... S_{i_l}^{\varepsilon_l}$.

Proposition 27. *For an integrable $U_q(\mathfrak{g})$-module V and $w \in \tilde{\mathbb{W}}$, one has $A_{w,V}^+ = (-1)^{\rho^\vee - w\rho^\vee} \mathbb{S}_w$, where $\rho^\vee$ is an element of $\mathfrak{h}$ such that $\alpha_i(\rho^\vee) = 1$ for all i.*

Proof. This is immediate from Proposition 25 and the definition. □

Remark. Analogously to this proposition, the limits $A_{w,V}^C$ of the operators $A_{w,V}(\lambda)$ in Weyl chambers C other than the dominant one give rise to other variants of the quantum Weyl group.

5.3. **The dynamical action of the pure braid group.** Define the **pure braid group** $P\tilde{\mathbb{W}}$ to be the kernel of the natural homomorphism $\tilde{\mathbb{W}} \to \mathbb{W}$. This group is the smallest normal subgroup of $\tilde{\mathbb{W}}$ which contains s_i^2 for all i. So it is generated by the elements $w s_i^2 w^{-1}, w \in \tilde{\mathbb{W}}$.

The dynamical action of $\tilde{\mathbb{W}}$ on functions of λ with values in an integrable $U_q(\mathfrak{g})$-module V induces an action of $P\tilde{\mathbb{W}}$ on functions with values in any weight subspace $V[\nu]$ of V, which is linear over the field F of scalar meromorphic functions of λ.

Proposition 28. *The group $P\tilde{\mathbb{W}}$ acts in $V[\nu]$ by multiplication by a character $\chi : P\tilde{\mathbb{W}} \to F^*$, which is $\tilde{\mathbb{W}}$-equivariant (i.e. $\chi(wpw^{-1})(\lambda) = \chi(p)(w\lambda)$), and is determined by the relations*

$$\chi(s_i^2) = (-1)^{\nu(h_i)} \frac{[(\lambda+\nu, \alpha_i)]_q}{[\lambda]_q},$$

Proof. It is sufficient to prove the result for $\mathfrak{sl}_2$. In this case, we have a unique simple reflection s, and

$$(s*)^2 = \mathcal{A}_V(-\lambda)\mathcal{A}_V(\lambda) = (A_V^\infty)^2 B_V(\lambda-1)B_V(-\lambda-1)_{V[\beta]}.$$

But $(A_V^\infty)^2 = (-1)^h$ on V. Therefore, by the explicit formula for B_V we have $(s*)^2 = (-1)^{\beta(h)} \frac{[\lambda+\beta]_q}{[\lambda]_q}$, as claimed. ☐

It is easy to see that if V is an integrable module over $U_q(\mathfrak{g})$ then the dynamical Weyl group operators preserve the space of functions on $\mathfrak{h}^*$ with values in the zero weight subspace $V[0]$. So let us consider the dynamical action of the braid group restricted to this space.

Corollary 29. *The dynamical action of the pure braid group $P\tilde{\mathbb{W}}$ on functions of λ with values in $V[0]$ is trivial. Therefore, the dynamical action of $\tilde{\mathbb{W}}$ on this space induces an action of the Weyl group $\mathbb{W}$. In particular, the 1-cocycle relation $A_{w_1w_2,V}(\lambda) = A_{w_1,V}(\lambda)A_{w_2,V}(w_1\cdot\lambda)$ of Lemma 17 is satisfied for any $w_1, w_2 \in \mathbb{W}$ (i.e. without the requirement $l(w_1w_2) = l(w_1) + l(w_2)$).*

Proof. Follows immediately from Proposition 28. ☐

Remark. Note that these results are false for the usual (non-dynamical) quantum Weyl group, unless $q = 1$. The action of the pure part of the quantum Weyl group does not, in general, reduce to a character, and the action on the zero weight subspace is, in general, nontrivial. This seemingly paradoxical situation is similar to the situation with trigonometric R-matrices with a spectral parameter: these R-matrices satisfy the involutivity condition and hence define representations of S_n, but tend at infinity to R-matrices without spectral parameter, which fail to satisfy involutivity and hence define only braid group representations.

5.4. **The operators $B^\pm_{w,V}(\lambda)$.** Let $w \in \mathbb{W}$. Following [TV], define the operators $B^\pm_{w,V}(\lambda)$ by the formula

$$B^\pm_{w,V}(\lambda) = (A^\pm_{w,V})^{-1}A_{w,V}(\lambda).$$

For example, in the $U_q(\mathfrak{sl}_2)$ case, and w being the only nontrivial element, we have $B^\pm_{w,V}(\lambda) = B^\pm_V(\lambda)$ (see the notation in Section 3).

Proposition 30. *(i) $B^{\pm}_{w,V}$ preserves the weight decomposition.*

(ii) We have $B^+_{w,V} \to 1$ as $q^{(\lambda,\alpha_i)} \to +\infty$; $B^-_{w,V} \to 1$ as $q^{(\lambda,\alpha_i)} \to 0$.

(iii) If $q=1$, then $B^+_{w,V}$ equals $B^-_{w,V}$.

Proof. The proof of this proposition is straightforward. □

Thus, if $q=1$, we will denote $B^+_{w,V} = B^-_{w,V}$ simply by $B_{w,V}$.

From the definition of the dynamical Weyl group it follows that the operators $B^{\pm}_{w,V}$, like the operators $A_{w,V}$, admit a factorization into a noncommutative product of $l(w)$ terms. For the sake of brevity, we will consider only the operators $B^+_{w,V}$; the story for $B^-_{w,V}$ is analogous.

Let $w \in \mathbb{W}$. Fix a reduced decomposition δ of w: $w = s_{i_1}...s_{i_l}$. Recall that in Section 4 we assigned to this reduced decomposition a sequence of roots $\alpha^1, ..., \alpha^l$.

It follows from Lusztig's theory of braid group actions on $U_q(\mathfrak{g})$ [Lu] that there exists a unique element $e^{\delta}_{\pm\alpha^j}$ of $U_q(\mathfrak{g})$ which acts in any integrable module V as

$$e^{\delta}_{\pm\alpha_j} = A^+_{s_{i_l}...s_{i_{j+1}}} e_{\pm\alpha_{i_j}} (A^+_{s_{i_l}...s_{i_{j+1}}})^{-1}$$

(a "Cartan-Weyl generator"). We also let $h_{w\alpha_i} = w(h_i)$.

Proposition 31. *The operator $B^+_{w,V}(\lambda)$ is obtained by the action in V of the universal element*

$$B^+_w(\lambda) = \prod_{j=1}^{l} p((\lambda+\rho)(h_{\alpha^j}) - 1, h_{\alpha^j}, e^{\delta}_{\alpha^j}, e^{\delta}_{-\alpha^j}),$$

where the indices increase from left to right.

Proof. Straightforward from the definition. □

In particular, Proposition 31 implies that the operator $B^+_{w,V}(\lambda)$, unlike the operator $A_{w,V}(\lambda)$, is well defined on weight spaces of any module V over $U_q(\mathfrak{g})$ from category $\mathcal{O}$ (not necessarily integrable). Namely, it is defined by the product formula of Proposition 31.

Proposition 31 also implies the following determinant formula for $B^+_{w,V}$ acting on a weight subspace.

Let $w \in \mathbb{W}$ and $w = s_{i_1}...s_{i_l}$ be a reduced decomposition of w. Let β be a weight of a finite dimensional $U_q(\mathfrak{g})$-module V. Let

$$B^+_{\alpha\beta k}(q,\lambda) = \prod_{j=1}^{k} \frac{q^{-2((\lambda+\rho,\alpha^\vee)+j)} - 1}{q^{-2((\lambda+\rho-\beta,\alpha^\vee)-j)} - 1}.$$

Corollary 32. *One has*

$$det(B^+_{w,V}(\lambda)|_{V[\beta]}) = \prod_{j=1}^{l} \prod_{k\geq 0} B^+_{\alpha^j\beta k}(q_{i_j},\lambda)^{d_V(\alpha^j,\beta,k)},$$

where

$$d_V(\alpha,\beta,k) = \dim(V[\beta+k\alpha]) - \dim(V[\beta+(k+1)\alpha]).$$

Proof. Straightforward from the definition. □

5.5. **The operators $B^+_{w_0,V}(\lambda)$ and extremal projectors.** Let $\mathfrak{g}$ be finite dimensional, w_0 the maximal element of the Weyl group of $\mathfrak{g}$. Let M_μ be the Verma module with highest weight μ. Let γ be a nonnegative linear combination of simple roots.

Proposition 33. *For generic μ, the operator-valued function $B^+_{w_0,M_\mu}(\lambda)|_{M_\mu[\mu-\gamma]}$ is regular at the point $\lambda=-2\rho$, and the operator $B^+_{w,M_\mu}(-2\rho)$ is the projector to the highest weight space of M_μ along the sum of other weight spaces (the extremal projector). In other words, $B^+_{w,M_\mu}(-2\rho)|_{M_\mu[\mu-\gamma]} = \delta_{\gamma 0} Id$.*

Proof. The first statement is immediate from Proposition 31, as for $\mathfrak{g} = \mathfrak{sl}_2$, the operator valued function $B^+(\lambda)$ of λ is regular at integer values of λ when restricted to a weight subspace of a generic Verma module.

Let us now prove the second statement. Because μ is generic, any nonzero homogeneous vector $v\in M_\mu$ of weight different from μ is not singular. Thus, it suffices to show that for any non-singular homogeneous vector v, one has $B^+_{w_0,M_\mu}v=0$.

Let v be a non-singular homogeneous vector in M_μ. Then there exists an index i such that $e_iv\neq 0$. We will assume that v is an eigenvector for the Casimir operator of $U_{q_i}(\mathfrak{sl}_2)$, since this does not cause a loss of generality. Let Y_v be the submodule of M_μ over $U_{q_i}(\mathfrak{sl}_2)$ generated by v. Then Y_v is a Verma module, and v is a nonzero homogeneous vector in Y_v which is not singular.

Let $w_0 = s_{i_1}...s_{i_l}$ be the reduced decomposition of w_0, such that $i_l = i$. By Proposition 31, to this decomposition there corresponds a factorization

$$B^+_{w_0,M_\mu}(-2\rho) = \Pi\cdot p(-2,h_i,e_i,f_i),$$

where Π is the product of the terms in the product formula for $B^+_{w_0,V}$ corresponding to all but the last factor in the reduced decomposition of w_0. Thus, it suffices to show that $p(-2,h_i,e_i,f_i)v=0$.

But this is immediate from the product formula for the operator B in Corollary 8: the first factor in this product has numerator $\lambda+2$, and so for $\lambda=-2$, the product vanishes whenever the set of indices over which the product is taken is nonempty. The proposition is proved. □

We note that extremal projectors appeared in [AST] and there is an extensive theory of them (see e.g. [Zh1, Zh2]). In particular, Proposition 33, in the case of finite dimensional Lie algebras and $q=1$, essentially appears in [Zh1, Zh2] (see Theorem 2 of [Zh2]).

5.6. **The dynamical Weyl group of a locally finite module.** Let V be an $\mathfrak{h}$-diagonalizable $U_q(\mathfrak{g})$-module. Recall that V is said to be locally finite if any vector of $v\in V$ generates a finite dimensional submodule over the quantum $U_{q_i}(\mathfrak{sl}_2)$ subalgebras corresponding to all simple roots.

Lemma 34. *Let $\mathfrak{g}$ be finite dimensional, and let V be a locally finite $U_q(\mathfrak{g})$-module. Then the submodule Y_v generated in V by any vector v is finite dimensional.*

Proof. It suffices to assume that v is homogeneous with respect to the weight decomposition.

Let $\{e_\alpha\}$ be the Cartan-Weyl generators of $U_q(\mathfrak{g})$ corresponding to some reduced decomposition of the maximal element of $\mathbb{W}$. By the PBW theorem (see [Sa]), the submodule $Y_v \subset V$ is given by $Y_v = \prod_\alpha \mathbb{C}[e_\alpha]v$, where $\mathbb{C}[e_\alpha]$ is the algebra of polynomials of e_α, and the product is taken over all roots in a suitable order. Since the product is finite, and V is a sum of finite dimensional $\mathbb{C}[e_\alpha]$-modules (because e_α is conjugate to some e_i under Lusztig's braid group action on $U_q(\mathfrak{g})$), we have that Y_v is finite dimensional. □

It is clear that the operators $A_{w,V}(\lambda)$ are well defined for any locally finite module V over $U_q(\mathfrak{g})$.

Therefore, for any such V we can define the operators $w\circ$, $w \in \tilde{\mathbb{W}}$, on V-valued functions of λ as before. The above lemma allows us to prove the following.

Corollary 35. *For any locally finite $U_q(\mathfrak{g})$-module V, the operators $w\circ$ define an action of $\tilde{\mathbb{W}}$.*

Proof. We have to check that the operators $s_i\circ$ satisfy the braid relations. This can be checked on finite dimensional Lie algebras of rank 2. But in this case, according to Lemma 34, everything reduces to the case when V is finite dimensional, where the statement is known. □

6. The dynamical Weyl group element corresponding to the maximal element w_0 of $\mathbb{W}$

6.1. **The expression for $A_{w_0,V}$ and $B^+_{w_0,V}$.** In this section we will study the operator $A_{w_0,V}$ for the maximal element w_0 of $\mathbb{W}$ in the case of finite dimensional Lie algebras. This operator is especially important because, as we will show below, it is closely related to the operator $Q_V(\lambda)$, which is (as we will show elsewhere) the matrix of inner products of trace functions, and therefore generalizes squared norms of Macdonald polynomials. The material of this section allows one to calculate explicitly the operator Q_V and its determinant (see below), and therefore prove a matrix analog of the Macdonald inner product identities.

So, let $\mathfrak{g}$ be finite dimensional, and let $w_0 \in \mathbb{W}$ be the maximal element. Let $\{x_i\}$ be an orthonormal basis of $\mathfrak{h}$. Let U, V be finite dimensional modules over $U_q(\mathfrak{g})$.

Lemma 36. *One has in $V \otimes U$:*

$$A^+_{w_0,V\otimes U} = \mathcal{R}_0^{21}(A^+_{w_0,V} \otimes A^+_{w_0,U}),$$

and

$$Ad(A^+_{w_0,V} \otimes A^+_{w_0,U})(\mathcal{R}) = q^{\sum x_i\otimes x_i}\mathcal{R}_0^{21} = q^{\sum x_i\otimes x_i}\mathcal{R}^{21}q^{-\sum x_i\otimes x_i}.$$

Proof. The statements are obtained from Lemma 18 and Corollary 19 by passing sending λ to infinity. Namely, the first equation follows from Lemma 18 and Theorem 50 of [EV1]. The second equation follows from Corollary 19 and Theorem 50 of [EV1]. $\square$

Remark. Since $A^+_{w,V}$ are, essentially, the quantum Weyl group operators, Lemma 36 can also be deduced from the theory of the quantum Weyl group (see [KoSo]). In fact, this lemma is nothing but the coproduct property of the maximal element, which is an important property of the quantum Weyl group. This property is instrumental in deriving the Levendorski-Soibelman product formula for the universal R-matrix ([KoSo]).

Let $Q^\dagger_V(\lambda) = Q_{V^*}(\lambda)^*$ (in the notation of [EV2], $Q^\dagger_V = S(Q)|_V$, where S is the antipode).

In this subsection we will prove the following result.

Theorem 37. *One has*

$$A_{w_0,V}(\lambda) = A^+_{w_0,V} Q^\dagger_V(\lambda).$$

In other words, $Q^\dagger_V(\lambda) = B^+_{w_0,V}(\lambda)$.

Remark. Note that in the formula for $Q^\dagger$ obtained from the definition, the f-terms come on the left from the e-terms, while in the product formula for $B^+_{w_0}$, the f-terms and the e-terms are mixed with each other. Thus, the theorem provides a way to perform a "normal ordering" of the terms in the product formula for $B^+_{w_0}$.

The proof of Theorem 37 occupies the rest of the subsection.

Consider the universal fusion matrix $J(\lambda)$, i.e. the element of the completed tensor square of $U_q(\mathfrak{g})$ which acts in the product $V \otimes U$ of finite dimensional $U_q(\mathfrak{g})$-modules as $J_{VU}(\lambda)$ (see [EV1]). Let $Q(\lambda) = \sum S^{-1}(c'_i)c_i$, where $J(\lambda) = \sum c_i \otimes c'_i$, and S is the antipode. Let $Q^\dagger(\lambda) = S(Q(\lambda))$. Then $Q|_V = Q_V$, $Q^\dagger|_V = Q^\dagger_V$.

It follows from formula (2.38) in [EV2] that

$$\Delta(Q(\lambda)) = (S \otimes S)(J^{21}(\lambda))^{-1} \cdot (Q(\lambda) \otimes Q(\lambda + h^{(1)})) \cdot J(\lambda + h^{(1)} + h^{(2)})^{-1}.$$

Applying the antipode to this equation and permuting the components, we obtain

$$\Delta(Q^\dagger(\lambda)) = (S \otimes S)(J^{21}(\lambda + h^{(1)} + h^{(2)}))^{-1} \cdot (Q^\dagger(\lambda - h^{(2)}) \otimes Q^\dagger(\lambda)) \cdot J(\lambda)^{-1}.$$

Here we use the fact that $S^2 = \mathrm{Ad}(q^{2\rho})$ and the zero weight property of J; note also that we replaced $h^{(i)}$ by $-h^{(i)}$ in the Q-terms of the equation, since the antipode changes the sign.

On the other hand, we have shown in Lemma 36 that

$$A^+_{w_0,V\otimes U} = \mathcal{R}_0^{21}(A^+_{w_0,V} \otimes A^+_{w_0,U}).$$

Therefore, we have the following lemma.

Lemma 38.

$$A^+_{w_0,V\otimes U} Q^\dagger_{V\otimes U}(\lambda) =$$
$$\mathcal{R}_0^{21} \cdot (A^+_{w_0,V} \otimes A^+_{w_0,U}) \cdot (S \otimes S)(J^{21}(\lambda + h^{(1)} + h^{(2)}))^{-1} \cdot (Q^\dagger(\lambda - h^{(2)}) \otimes Q^\dagger(\lambda)) \cdot J(\lambda)^{-1}.$$

We will also need the following lemma.

Lemma 39. *One has on $V \otimes U$:*

$$\mathcal{R}_0^{21} \cdot Ad(A^+_{w_0,V} \otimes A^+_{w_0,U}) \cdot \left((S \otimes S)(J^{21}(\lambda + h^{(1)} + h^{(2)}))^{-1}\right) = J(w_0 \cdot \lambda).$$

Proof. We will first transform the equality to a convenient form, and then show that both sides satisfy the same ABRR equation ([ABRR, ES]), which has a unique solution. This will imply that the two sides are equal.

Let us make a change of variable $\mathbb{J}(\lambda) = J(-\lambda - \rho)$. Then the equation to be proved takes the form

$$\mathcal{R}_0^{21} \cdot \mathrm{Ad}(A^+_{w_0,V} \otimes A^+_{w_0,U}) \cdot \left((S \otimes S)(\mathbb{J}^{21}(\lambda - h^{(1)} - h^{(2)}))^{-1}\right) = \mathbb{J}(w_0\lambda).$$

Let $\mathcal{J}(\lambda) = \mathbb{J}(\lambda - \frac{1}{2}(h^{(1)} + h^{(2)}))$. Then the equation takes the form

$$\mathcal{R}_0^{21} \cdot \mathrm{Ad}(A^+_{w_0,V} \otimes A^+_{w_0,U}) \cdot \left((S \otimes S)(\mathcal{J}^{21})^{-1}\right)(\lambda + \frac{1}{2}w_0(h^{(1)} + h^{(2)})) = \mathcal{J}(w_0\lambda + \frac{1}{2}(h^{(1)} + h^{(2)})).$$

Replacing $\lambda + \frac{1}{2}w_0(h^{(1)} + h^{(2)})$ with λ, we obtain the equation

$$\mathcal{R}_0^{21} \cdot \mathrm{Ad}(A^+_{w_0,V} \otimes A^+_{w_0,U}) \cdot \left((S \otimes S)(\mathcal{J}^{21}(\lambda))^{-1}\right) = \mathcal{J}(w_0\lambda).$$

To establish this equation, let us recall that by Lemma 2.4 of [EV2] (see also [ABRR]), the element $X(\lambda) = \mathcal{J}(\lambda)$ satisfies the ABRR equation

$$\mathcal{R}^{21}(q^{2\lambda})_1 X(\lambda) = X(\lambda) q^{\sum x_i \otimes x_i}(q^{2\lambda})_1.$$

Therefore, the element $Y(\lambda) = (S \otimes S)(\mathcal{J}^{21}(\lambda))^{-1}$ satisfies the equation

$$\mathcal{R}^{-1}(q^{2\lambda})_2 Y(\lambda) = Y(\lambda) q^{-\sum x_i \otimes x_i}(q^{2\lambda})_2.$$

Thus, using Lemma 36, we get that the operator on $V \otimes U$ given by $Z(\lambda) = \mathrm{Ad}(A^+_{w_0,V} \otimes A^+_{w_0,U})(Y(\lambda))$ satisfies the equation

$$(\mathcal{R}_0^{21})^{-1} q^{-\sum x_i \otimes x_i}(q^{2w_0\lambda})_2 Z(\lambda) = Z(\lambda) q^{-\sum x_i \otimes x_i}(q^{2w_0\lambda})_2.$$

Therefore, the operator $T(\lambda) = \mathcal{R}_0^{21} Z(\lambda)$ satisfies

$$(q^{2w_0\lambda})_2 \mathcal{R}^{21^{-1}} T(\lambda) = T(\lambda) q^{-\sum x_i \otimes x_i}(q^{2w_0\lambda})_2.$$

Transforming this using the weight zero property of T, we get

$$\mathcal{R}^{21}(q^{2w_0\lambda})_1 T(\lambda) = T(\lambda)(q^{2w_0\lambda})_1 q^{\sum x_i \otimes x_i}.$$

Now we note that the same equation is satisfied by $\mathcal{J}(w_0\lambda)$, by Lemma 2.4 of [EV2]. Both of these solutions are triangular, with the diagonal part equal to 1. But Lemma 2.4 of [EV1] claims that such a solution is unique.

Therefore, $T(\lambda) = \mathcal{J}(\lambda)$, and the lemma is proved. ☐

Corollary 40. *The operators $E_V(\lambda) := A_{w_0,V}(\lambda)^{-1} A^+_{w_0,V} Q^\dagger_V(\lambda)$ have the "grouplike" property*

$$E_{V \otimes U}(\lambda) = J(\lambda)(E_V(\lambda - h^{(2)}) \otimes E_U(\lambda))J^{-1}(\lambda).$$

Proof. This follows directly from Lemma 18, Lemma 38, and Lemma 39. ☐

Now we can prove the theorem. Let v be a highest weight vector of V. It is easy to see that in this case $Q_V^\dagger(\lambda)v = v$ (this follows from triangularity of J), and $A_{w_0,V}(\lambda)v$ is constant. This implies that $E_V(\lambda)v = v$. But Part (i) of Lemma 2.15 in [EV2] says that a collection E_V with the group-like property such that $E_V v = v$ on highest weight vectors must necessarily be trivial: $E_V = 1$. The theorem is proved.

Remark. Here is a proof of Corollary 14 using Theorem 37, different from the proof in [TV]. By Theorem 37, $B^+(\lambda) = Q^\dagger(\lambda)$. Adapting the formula of [BBB] for the universal fusion matrix to our conventions, we get that the element $\mathcal{J}(\lambda)$ has the form

$$\mathcal{J}(\lambda) = \sum_{k=0}^{\infty} q^{-k(k+1)/2} \frac{(1-q^2)^k}{[k]_q!} (f^n \otimes e^n) \prod_{\nu=1}^{n} \frac{q^{2\lambda}}{1 - q^{2\lambda+2\nu}(q^{-h} \otimes q^h)}.$$

Applying the antipode to the first component, multiplying the components, and changing λ to $-\lambda - 1$, we get the result.

6.2. **The determinant of $Q^\dagger$.** Theorem 37 allows us to compute explicitly the determinant of $Q^\dagger$ on every weight subspace of a finite dimensional module.

Proposition 41. *One has*

$$det(Q^\dagger(\lambda)|_{V[\beta]}) = \prod_{\alpha>0} \prod_{k\geq 0} B^+_{\alpha\beta k}(q_\alpha, \lambda)^{d_V(\alpha,\beta,k)},$$

where

$$d_V(\alpha, \beta, k) = \dim(V[\beta + k\alpha]) - \dim(V[\beta + (k+1)\alpha]).$$

Proof. The proposition is immediate from Theorem 37 and Proposition 32. ☐

7. Applications of the dynamical Weyl groups to trace functions

In this section we will assume for simplicity that $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, although the results can be generalized to Kac-Moody algebras (see [ES2]).

7.1. **A generalized Weyl character formula.** Recall that in [EV2] we defined the trace functions

$$\Psi^v(\lambda, \mu) = \mathrm{Tr}|_{M_\mu}(\Phi^v_\mu q^{2\lambda}),$$
$$\Psi_V(\lambda, \mu) = \sum \Psi^{v_i}(\lambda, \mu) \otimes v_i^* \in V[0] \otimes V^*[0],$$

where v_i is a basis of $V[0]$ and v_i^* is the dual basis of $V^*[0]$.

Suppose μ is a large dominant integral weight, and L_μ is the irreducible finite dimensional representation with this highest weight. The intertwining operator Φ^v_μ descends to an operator $\bar{\Phi}^v_\mu : L_\mu \to L_\mu \otimes V$, and we define

$$\Psi^v_\mu(\lambda) = \mathrm{Tr}|_{L_\mu}(\bar{\Phi}^v_\mu q^{2\lambda}),$$
$$\Psi^\mu_V(\lambda) := \sum \Psi^{v_i}_\mu(\lambda) \otimes v_i^*.$$

Let us regard $\Psi_V(\lambda,\mu)$, $\Psi_V^\mu(\lambda)$ as linear operators on $V[0]$.

Proposition 42. *One has*

$$\Psi_V^\mu(\lambda)=\sum_{w\in\mathbb{W}}(-1)^w\Psi_V(\lambda,w\cdot\mu)A_{w,V}(\mu).$$

Proof. The proof is analogous to the proof of the Weyl character formula using the approach of [BGG]; it is based on the fact that in the Grothendieck group of the category $\mathcal{O}$, an irreducible module is an alternating sum of Verma modules. ☐

Remark 1. If $V=\mathbb{C}$, this formula reduces to the Weyl character formula.

Remark 2. If $\mathfrak{g}=\mathfrak{sl}_n$, $V=S^{rn}\mathbb{C}^n$ (the "Macdonald" case), then $V[0]$ is 1-dimensional, and so the action of $A_{w,V}$ on $V[0]$ can be computed explicitly using the expression for A_{s,V_m} for $\mathfrak{sl}_2$. In this case it is easy to show that Proposition 42 reduces to Conjecture 8.2 in [FV], which was proved in [ESt] (Prop. 5.3).

Recall that in [EV2] we also defined renormalized trace functions

$$F_V(\lambda,\mu)=\delta_q(\lambda)\Psi_V(\lambda,-\mu-\rho)Q_V^{-1}(-\mu-\rho),$$

where $\delta_q(\lambda)$ is the Weyl denominator: $\delta_q(\lambda)=\sum_{w\in\mathbb{W}}(-1)^w q^{2(\lambda,w\rho)}$.

Let us say that a weight μ is antidominant if $-\mu$ is dominant. By analogy with the above, one can also define for large antidominant integral weight μ

$$F_V^\mu(\lambda)=\delta_q(\lambda)\Psi_V^{-\mu-\rho}(\lambda)Q_V^{-1}(-\mu-\rho),$$

Corollary 43. *For an antidomonant μ with sufficiently large coordinates one has*

$$F_V^\mu(\lambda)=\sum_{w\in\mathbb{W}}(-1)^wF_V(\lambda,w\mu)(\mathcal{A}_{w,V^*}(\mu)^{-1})^*.$$

Proof. The proof follows from Proposition 20, and Proposition 42. ☐

7.2. **Dynamical Weyl group invariance of Macdonald-Ruijsenaars operators and renormalized trace functions.** Let $q\neq 1$. Recall that in [EV2] we defined the modified dynamical R-matrix $\mathbb{R}_{UV}(\lambda)=R_{UV}(-\lambda-\rho)$, and introduced Macdonald-Ruijsenaars operators, acting on rational functions of λ with values in $V[0]$

$$\mathcal{D}_U=\sum_\nu \mathrm{Tr}|_{U[\nu]}(\mathbb{R}_{UV}(\lambda))T_\nu,$$

where T_ν maps $f(\lambda)$ to $f(\lambda+\nu)$.

Proposition 44. *The Macdonald-Ruijsenaars operators are invariant with respect to the dynamical action of $\mathbb{W}$ on functions on $\mathfrak{h}^*$ with values in $V[0]$. That is, $[\mathcal{D}_U,w*]=0$ for $w\in\mathbb{W}$.*

Proof. We have

$$((w*)^{-1}\mathcal{D}_U(w*)f)(\lambda) = \mathcal{A}_w(\lambda)^{-1}\sum_\nu \mathrm{Tr}|_{U[w\nu]}(\mathbb{R}(w\lambda))\mathcal{A}_w(\lambda+\nu)f(\lambda+\nu)$$

(for brevity we drop the subscripts indicating the modules in which the operators act). From Corollary 19 we easily obtain

$$\mathbb{R}(w\lambda) = \mathcal{A}_w^{(2)}(\lambda)\mathcal{A}_w^{(1)}(\lambda+h^{(2)})\mathbb{R}_{VU}(\lambda)\mathcal{A}_w^{(2)}(\lambda+h^{(1)})^{-1}\mathcal{A}_w^{(1)}(\lambda)^{-1}$$

Let us substitute this equation into the previous equation, and use the fact that in the second component we are restricting to the zero-weight subspace. It is easy to see that the $\mathcal{A}$-factors cancel, and we get

$$(w*)^{-1}\mathcal{D}_U(w*) = \mathcal{D}_U,$$

as desired. □

Let us return to the renormalized trace function $F_V(\lambda,\mu)$, which we will now regard as an element of $V[0]\otimes V^*[0]$.

Recall from [EV2] that $F_V(\lambda,\mu)$ satisfies the Macdonald-Ruijsenaars equations

$$\mathcal{D}_U^{(\lambda)}F_V(\lambda,\mu) = \chi_U(q^{-2\mu})F_V(\lambda,\mu),$$

the dual Macdonald-Ruijsenaars equations

$$\mathcal{D}_U^{(\mu)}F_V(\lambda,\mu) = \chi_U(q^{-2\lambda})F_V(\lambda,\mu),$$

and has the symmetry property $F_V(\lambda,\mu) = F_{V^*}(\mu,\lambda)^*$ (here the superscripts λ,μ denote the variables with respect to which to take shifts when applying difference operators, and $()^*$ denotes the operator of exchanging the factors $V[0]$ and $V^*[0]$).

Proposition 45. *The function $F_V(\lambda,\mu)$ is invariant under the dynamical action of $\mathbb{W}$ on functions with values in $V[0]$. That is,*

$$F_V(\lambda,\mu) = (\mathcal{A}_{w,V}(w^{-1}\lambda)\otimes\mathcal{A}_{w,V^*}(w^{-1}\mu))F_V(w^{-1}\lambda,w^{-1}\mu),\ w\in\mathbb{W}\,.$$

Proof. The proof is similar to the proof of the symmetry of F_V, given in [EV2].

It suffices to assume that $q\neq 1$. Let $F'_V(\lambda,\mu)$ denote the right hand side of the equality to be proved. By Proposition 44, F'_V, like F_V, is a solution of the Macdonald-Ruijsenaars equations and the dual Macdonald-Ruijsenaars equations. Moreover, both F_V, F'_V have the form: $q^{2(\lambda,\mu)}$ times a finite sum of rational functions of $q^{(\lambda,\alpha_i)}$ multiplied by rational functions of $q^{(\mu,\alpha_i)}$ (where the denominators of the rational functions are products of binomials of the form $1-q^{(\lambda,\beta)}$, respectively $1-q^{(\mu,\beta)}$).

Let us regard F_V, F'_V as functions with values in $\mathrm{End}(V[0])$. It is easy to see, using power series expansions, that a solution of the Macdonald-Ruijsenaars equations with the above properties is unique up to right multiplication by an operator depending rationally of $q^{(\mu,\alpha_i)}$. Similarly, a solution of the dual Macdonald-Ruijsenaars equations with such

properties is unique up to left multiplication by an operator depending rationally on $q^{(\lambda,\alpha_i)}$. So we have

$$F'_V(\lambda,\mu) = X(\lambda)F_V(\lambda,\mu), \qquad F'_V(\lambda,\mu) = F_V(\lambda,\mu)Y(\mu),$$

where X, Y are rational operator valued functions of $q^{(\lambda,\alpha_i)}, q^{(\mu,\alpha_i)}$, and hence

$$X(\lambda)F_V(\lambda,\mu) = F_V(\lambda,\mu)Y(\mu).$$

Let us take the limit $q^{(\lambda,\alpha_i)} \to 0$ (for all i) in the last equality. It follows from the asymptotics of intertwiners (see [ESt]) that in this limit F_V is equivalent to $q^{-(\lambda,\mu)}\mathrm{Id}$. So we get $\lim X(\lambda) = Y(\mu)$ for all μ. Thus, $Y(\mu)$ is a constant operator. Using the symmetry of F_V, we get that $X(\lambda)$ is also a constant, so we get $X(\lambda) = Y(\mu) = X$, where X is a constant operator.

Finally, let us show that $X = 1$. We have the identity

$$XF_V(\lambda,\mu) = \mathcal{A}_{w,V}(w^{-1}\lambda)F_V(w^{-1}\lambda, w^{-1}\mu)\mathcal{A}_{w,V^*}(w^{-1}\mu)^*.$$

Using Proposition 21, we can rewrite this equation in the form

$$XF_V(\lambda,\mu) = \mathcal{A}_{w,V}(w^{-1}\lambda)F_V(w^{-1}\lambda, w^{-1}\mu)\mathcal{A}_{w^{-1},V}(-\mu).$$

Now let us take the limit: $q^{(\mu,\alpha_i)} \to 0$, $q^{(w^{-1}\lambda,\alpha_i)} \to 0$. Then, using Corollary 24, we get

$$X = A^-_{w,V}A^+_{w^{-1},V} = 1,$$

as desired. □

7.3. **The multicomponent dynamical action and invariance of multicomponent trace functions.** Let $V_1, ..., V_N$ be integrable $U_q(\mathfrak{g})$-modules. Define the linear operator $A_{w,V_1,...,V_n}(\lambda) : V_1 \otimes ... \otimes V_N \to V_1 \otimes ... \otimes V_N$ by

$$A_{w,V_1,...,V_N}(\lambda) = A^{(N)}_{w,V_N}(\lambda)A^{(N-1)}_{w,V_{N-1}}(\lambda - h^{(N)})...A^{(1)}_{w,V_1}(\lambda - h^{(2)} - ... - h^{(N)}).$$

With these operators one can associate the action of $\tilde{\mathbb{W}}$ on functions of λ with values in $V_1 \otimes ... \otimes V_N$ given by

$$(w \bullet f)(\lambda) = A_{w,V_1,...,V_N}(w^{-1}\cdot\lambda)f(w^{-1}\cdot\lambda).$$

We call this action the **shifted multicomponent dynamical action**. As before, the (unshifted) **multicomponent dynamical action** is defined by

$$w \diamond f = \mathcal{A}_{w,V_1,...,V_N}(w^{-1}\lambda)f(w^{-1}\lambda),$$

where

$$\mathcal{A}_{w,V_1,...,V_N}(\lambda) = A_{w,V_1,...,V_N}\left(-\lambda - \rho + \frac{1}{2}(\sum_{i=1}^N h^{(i)})\right)$$

Recall from [EV2] that the operator $J^{1...N}(\lambda)$ on $V_1 \otimes ... \otimes V_N$ is defined by

$$J^{1...N}(\lambda) = J^{1,2...N}(\lambda)...J^{N-1,N}(\lambda).$$

Lemma 46. *Conjugation with the operator $J^{1...N}$ transforms the shifted dynamical action of $\tilde{\mathbb{W}}$ into its shifted multicomponent dynamical action. That is,*

$$J^{1...N}(w\bullet) = (w\circ)J^{1...N}.$$

Proof. This follows by applying Lemma 18 several times. □

Recall from [EV2] the definition of the quantum KZB operators $K_j, K_j^\vee$ and the diagonal operators $D_j, D_j^\vee$ acting on functions of λ and μ with values in $(V_1 \otimes ... \otimes V_N)[0]$, respectively $(V_N^* \otimes ... \otimes V_1^*)[0]$ $(j = 1, ..., N)$. Namely, set

$$D_j = (q^{-2\mu - \sum x_i^2})_{*j}(q^{-2\sum x_i \otimes x_i})_{*j,*1...*j-1},$$

(where $*i$ labels the component V_i^*), and

$$K_j = \mathbb{R}_{j+1,j}(\lambda + h^{(j+2...N)})^{-1} ... \mathbb{R}_{Nj}(\lambda)^{-1}\Gamma_j \times$$
$$\mathbb{R}_{j1}(\lambda + h^{(2...j-1)} + h^{(j+1...N)}) ... \mathbb{R}_{jj-1}(\lambda + h^{(j+1...N)}),$$

where $\Gamma_j f(\lambda) := f(\lambda + h^{(j)})$, and $h^{j...k}$ acting on a homogeneous multivector has to be replaced with the sum of weights of components $j, ..., k$ of this multivector. Analogously, define the operators

$$D_j^\vee = (q^{-2\lambda - \sum x_i^2})_j (q^{-2\sum x_i \otimes x_i})_{j,j+1...N},$$

and

$$K_j^\vee = \mathbb{R}_{*j-1,*j}(\mu + h^{(*1...*j-2)})^{-1} ... \mathbb{R}_{*1,*j}(\mu)^{-1}\Gamma_{*j} \times$$
$$\mathbb{R}_{*j,*N}(\mu + h^{(*j+1...*N-1)} + h^{(*1...*j-1)}) ... \mathbb{R}_{*j,*j+1}(\mu + h^{(*1...*j-1)}),$$

where $\Gamma_{*j} f(\mu) = f(\mu + h^{*j})$.

In [EV2], we defined the multicomponent renormalized trace functions $F_{V_1,...,V_N}(\lambda, \mu)$ with values in $(V_1 \otimes ... \otimes V_N)[0] \otimes (V_N^* \otimes ... \otimes V_1^*)[0]$. Two of our main results were the identities

$$(K_j \otimes D_j)F_{V_1,...,V_N} = F_{V_1,...,V_N}$$

(the quantum KZB equations), and

$$(D_j^\vee \otimes K_j^\vee)F_{V_1,...,V_N} = F_{V_1,...,V_N}$$

(the dual quantum KZB equations).

Corollary 47. *(i) The multicomponent renormalized trace functions $F_{V_1,...,V_N}(\lambda, \mu)$ are invariant under the multicomponent dynamical action $w\diamond$ of $\mathbb{W}$ in both components. That is:*

$$((w\diamond)^{(\lambda)} \otimes (w\diamond)^{(\mu)})F_{V_1,...,V_N}(\lambda, \mu) = F_{V_1,...,V_N}(\lambda, \mu).$$

(ii) The quantum KZB operators $K_j, K_j^\vee$ and the diagonal operators $D_j, D_j^\vee$ are invariant under the multicomponent dynamical action of $\mathbb{W}$. In particular, the qKZB and dual qKZB equations are invariant under the multicomponent dynamical action.

Proof. Statement (i) follows from Lemma 46, Proposition 45 and the definitions of [EV2]. Statement (ii) can be checked directly using Corollary 19. □

8. Dynamical Weyl groups for affine Lie algebras and quantum affine algebras

In this section we will consider the dynamical Weyl group in the case when the role of $\mathfrak{g}$ is played by an affine Kac-Moody Lie algebra $\tilde{\mathfrak{g}}$, and the role of integrable representations V of $\mathfrak{g}$ or $U_q(\mathfrak{g})$ is played by representations on Laurent polynomials in one variable with coefficients in a finite dimensional vector space (we call them loop representations).

This situation turns out to be especially interesting for applications. Although this setting is very similar to the one already considered, it is not exactly the same, since loop representations are not integrable. Therefore, we will describe the changes that are necessary to carry out our main construction in this new situation.

8.1. **Affine algebras and loop representations.** In this subsection we will recall some standard facts about finite dimensional representations of classical and quantum affine algebras. The material on the classical affine algebras is standard; most of the material on the quantum case can be found in the book [CP], and references therein.

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. Let $(,)$ be the form on $\mathfrak{g}$ defined in Section 2, and let the positive integer m be defined by $(\theta,\theta)=2m$ for the maximal root θ of $\mathfrak{g}$ (we have $m=1$ in the simply laced case, $m=3$ for $\mathfrak{g}=G_2$, and $m=2$ otherwise). Let $\hat{\mathfrak{g}}=\mathfrak{g}[z,z^{-1}]\oplus\mathbb{C}c$ be the standard central extension of the loop algebra:

$$[a(z),b(z)]=[ab](z)+m\mathrm{Res}_0(a'(z),b(z))c.$$

Let $\tilde{\mathfrak{g}}=\mathbb{C}d\oplus\hat{\mathfrak{g}}$ be the extension of $\hat{\mathfrak{g}}$ by the scaling element d such that $[d,a(z)]=za'(z)$, $[d,c]=0$.

The Lie algebra $\tilde{\mathfrak{g}}$ is the Kac-Moody Lie algebra corresponding to the affine (i.e. extended) Cartan matrix of $\mathfrak{g}$. In particular, we have $\tilde{\mathfrak{g}}=\tilde{\mathfrak{h}}\oplus\hat{\mathfrak{n}}_+\oplus\hat{\mathfrak{n}}_-$, where $\hat{\mathfrak{n}}_\pm$ are the positive and negative nilpotent subalgebras, $\tilde{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d$. The restriction of the invariant bilinear form on $\tilde{\mathfrak{g}}$ to $\tilde{\mathfrak{h}}$ is defined by $(d,d)=(c,c)=0$, $(d,c)=1/m$, $(d,h)=(c,h)=0$, $h\in\mathfrak{h}$ (and the form on $\mathfrak{h}$ is the same as in $\mathfrak{g}$).

Remark. This normalization of the invariant form is traditional in the theory of quantum groups. On the other hand, the traditional bilinear form in the representation theory of affine Lie algebras and KZ equations is defined by the condition $(\theta,\theta)=2$ on the dual space, so it is m times bigger than our form. This is why many of our formulas have an extra factor m compared to the formulas from other texts about KZ equations (e.g. [EFK]).

The dual Cartan subalgebra of $\tilde{\mathfrak{g}}$ can be written as $\tilde{\mathfrak{h}}^*=\mathfrak{h}^*\oplus\mathbb{C}c^*\oplus\mathbb{C}d^*$, where c^*,d^* are the dual elements to c,d. Thus, elements of $\tilde{\mathfrak{h}}^*$ can be written as triples (λ,k,Δ), where $\lambda\in\mathfrak{h}^*$, $k,\Delta\in\mathbb{C}$, i.e. $(\lambda,k,\Delta)(h+ac+bd)=\lambda(h)+ka+\Delta b$. The number k is called the central charge of $\tilde{\lambda}$. For instance, the roots of $\tilde{\mathfrak{g}}$ are the elements of the form $(\alpha,0,n)$, where $n\in\mathbb{Z}$, α is 0 or is a root of $\mathfrak{g}$, and $(\alpha,n)\neq(0,0)$. The special weight $\rho_{\tilde{\mathfrak{g}}}$ for the affine algebra $\tilde{\mathfrak{g}}$ will be denoted by $\tilde{\rho}$. It has the form $\tilde{\rho}=\rho+h^\vee c^*$, where $h^\vee$

is the dual Coxeter number of $\mathfrak{g}$, and ρ is the special weight for the finite dimensional Lie algebra $\mathfrak{g}$, regarded as an element of $\tilde{\mathfrak{h}}^*$. In other words, $\tilde{\rho} = (\rho, h^\vee, 0)$.

We will denote the Chevalley generators of $\mathfrak{g}$ by e_i, f_i, h_i, $i = 1, ..., r$, and the additional generators of $\hat{\mathfrak{g}}$ by e_0, f_0, h_0. Similarly, $\alpha_1, ..., \alpha_r$ will stand for simple roots of $\mathfrak{g}$, and α_0 for the additional simple root of $\hat{\mathfrak{g}}$.

Let $U_q(\tilde{\mathfrak{g}})$, $U_q(\hat{\mathfrak{g}})$, $U_q(\hat{\mathfrak{n}}_\pm)$ be the quantum deformations of the corresponding classical objects, defined as in the general Kac-Moody case. In particular, the algebra $U_q(\tilde{\mathfrak{g}})$ contains elements q^{bc}, q^{bd}, $b \in \mathbb{C}$.

As before, we will consider the quantum situation but will allow q to be 1, unless otherwise specified.

Let us define the notion of a **loop representation** of $U_q(\tilde{\mathfrak{g}})$. Let $\bar{V}$ be a finite dimensional representation of $U_q(\hat{\mathfrak{g}})$. Set $V = \bar{V}[z, z^{-1}]$, with the following action of $U_q(\tilde{\mathfrak{g}})$: $d|_V = z\frac{d}{dz}$, and $x|_V = z^n x|_{\bar{V}}$ for $x \in U_q(\hat{\mathfrak{g}})$, such that $[d, x] = nx$.
Definition: V is called a loop representation.

Thus every loop representation V has a natural structure of a module over $\mathbb{C}[z, z^{-1}]$, and the underlying representation $\bar{V}$ is reconstructed by $\bar{V} = V/(z-1)V$. Moreover, if $a \in \mathbb{C}^*$ then we get a new finite dimensional representation $\bar{V}(a) = V/(z-a)V$ of $U_q(\hat{\mathfrak{g}})$. This representation is called the shift of $\bar{V}$ by a.

We will need the following proposition.

Proposition 48. *(i) Any finite dimensional representation Y of $U_q(\hat{\mathfrak{g}})$ has a weight decomposition with respect to $U_q(\hat{\mathfrak{h}}) \subset U_q(\hat{\mathfrak{g}})$ (where $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c$).*

(ii) In any finite dimensional representation Y of $U_q(\hat{\mathfrak{g}})$, the element c acts by zero (in the q-case, by this we mean that $q^{bc} = 1$).

(iii) Statements (i) and (ii) are valid for loop representations.

Proof. This Proposition is well known, but we will give a proof for the reader's convenience.

Statement (i) follows from the fact that any finite dimensional representation has a weight decomposition with respect to any (quantum) $\mathfrak{sl}_2$-subalgebra corresponding to a simple root (by representation theory of quantum $\mathfrak{sl}_2$).

Let us prove (ii). By the existence of a weight decomposition, it suffices to prove this for irreducible representations. But in an irreducible representation, c (respectively, q^{bc}) acts by a scalar. If $q = 1$, this scalar must be zero, as c is a linear combination of $[e_i, f_i]$ and thus $\mathrm{Tr}(c) = 0$. If $q \neq 1$, it suffices to show that $q^c = 1$ (as the weights are integral). But $(q_0 - q_0^{-1})[e_0, f_0] = q^{c-\theta^\vee} - q^{-c+\theta^\vee}$. Since $w_0\theta^\vee = -\theta^\vee$, in a finite dimensional $U_q(\mathfrak{g})$-module we have $Tr(q^{\theta^\vee}) = Tr(q^{-\theta^\vee}) \neq 0$. Thus,

$$0 = (q_0 - q_0^{-1})Tr([e_0, f_0]) = (q^c - q^{-c})Tr(q^{\theta^\vee}).$$

Thus, $q^c = \pm 1$, but it is an integer power of q, so $q^c = 1$ as desired.

Statement (iii) is clear from (i),(ii). □

The main examples of finite dimensional representations of $U_q(\hat{\mathfrak{g}})$ are the so called evaluation modules. To define them, let us first assume that $q = 1$. In this case, for any $a \in \mathbb{C}^*$, we have the evaluation homomorphism $\mathrm{ev}_a : U(\hat{\mathfrak{g}}) \to U(\mathfrak{g})$, defined by $\mathrm{ev}_a(x(z)) = x(a), \mathrm{ev}_a(c) = 0$. Let Y be a finite dimensional irreducible $\mathfrak{g}$-module. Then let $Y(a)$ denote the $\hat{\mathfrak{g}}$-module $\mathrm{ev}_a^* Y$ (the pullback of Y). This module is called *an evaluation module*. It is easy to see that the associated loop representation is $Y[z, z^{-1}]$, with pointwise action of $\hat{\mathfrak{g}}$.

For $q \neq 1$, by evaluation modules over $U_q(\hat{\mathfrak{g}})$ we will mean q-deformations of evaluation modules for $\hat{\mathfrak{g}}$; in other words, finite dimensional modules which remain irreducible when restricting to $U_q(\mathfrak{g})$.

Remark. For $\mathfrak{g} = \mathfrak{s}l_n$, there exists an analog of the homomorphism ev_a, introduced by Jimbo (see e.g. [EFK]). In this case, we can define the evaluation module $Y(a)$ corresponding to any irreducible finite dimensional $U_q(\mathfrak{g})$-module Y, in the same way as in the classical case. In other words, we can q-deform every evaluation module. Outside of type A, the evaluation homomorphism does not exist, and, as a result, not every evaluation module can be deformed (e.g. the module corresponding to the adjoint representation of $\mathfrak{g}$ cannot); but some evaluation modules can still be deformed, e.g. the vector representation for classical groups.

8.2. **The affine Weyl group.** In this subsection we will recall basic facts about affine Weyl groups. These facts are standard, and can be found in the literature (e.g. [Ch1, Ch2] and references therein), but we will give the definitions, statements, and even some proofs for the reader's convenience.

Let $\mathbb{W}^a$ denote the Weyl group of $\tilde{\mathfrak{g}}$. It has generators $s_0, ..., s_r$, satisfying the usual braid and involutivity relations.

Let $Q^\vee$ be the dual root lattice of $\mathfrak{g}$. It is well known that $\mathbb{W}^a$ is isomorphic to the semidirect product $\mathbb{W} \ltimes Q^\vee$ of the Weyl group $\mathbb{W}$ of $\mathfrak{g}$ with the dual root lattice $Q^\vee$ (i.e. the Cartesian product $\mathbb{W} \times Q^\vee$ with the product $(w, q)(w', q') = (ww', (w')^{-1}(q) + q')$), via the isomorphism defined by $s_i \to (s_i, 0), i \neq 0; s_0 \to (s_\theta, -\theta^\vee)$. In particular, $\mathbb{W}$ and $Q^\vee$ are subgroups of $\mathbb{W}^a$ in a natural way. To avoid confusion, given an element $\beta \in Q^\vee$, we will write t_β for the corresponding element of $\mathbb{W}^a$, and use the multiplicative (rather than the additive) notation for the product of such elements; thus, $t_\mu t_\nu = t_{\mu+\nu}$.

Let us compute the action of $\mathbb{W}^a$ on $\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}c^* \oplus \mathbb{C}d^*$. The action of $\mathbb{W}$ is obvious (i.e. it acts only on the $\mathfrak{h}^*$-component, keeping the other two unchanged), so let us calculate the action of the lattice $Q^\vee$.

Recall that elements of $\tilde{\mathfrak{h}}^*$ can be written as triples (λ, k, Δ).

We have the following result.

Lemma 49. *One has*

$$t_\nu(\lambda, k, \Delta) = (\lambda + mk\nu, k, \Delta - (\lambda, \nu) - mk(\nu, \nu)/2).$$

Proof. Let us call the operator defined by the right hand side by t'_ν. Using the identity $s_0 s_\theta = t_{\theta^\vee}$, we get that $t_\nu = t'_\nu$ for $\nu = \theta^\vee$. Since the statement that $t_\nu = t'_\nu$ is Weyl group invariant, and $t'_{\nu_1+\nu_2} = t'_{\nu_1} t'_{\nu_2}$, this is sufficient. $\square$

It is also useful to introduce the extended affine Weyl group $\mathbb{W}^b$. By the definition, $\mathbb{W}^b = \mathbb{W} \ltimes P^\vee$, where $P^\vee$ is the dual weight lattice. Thus, $\mathbb{W}^b$ naturally contains $\mathbb{W}^a$ as a subgroup. We can define the action of $\mathbb{W}^b$ on weights by extending the formula of Lemma 49 to elements $\nu \in P^\vee$. It is easy to check that the set of roots is invariant under $\mathbb{W}^b$ (as the pairing between $P^\vee$ and the root lattice Q takes only integer values: these two lattices are dual to each other).

Let G be the simply connected Lie group corresponding to $\mathfrak{g}$. It is easy to see that the exponential map $\varepsilon := \exp(2\pi i \cdot *) : \mathfrak{g} \to G$ identifies the group $P^\vee/Q^\vee = \Pi$ with the center G. Indeed, the lattice $Q^\vee$ is the kernel of ε restricted to $\mathfrak{h}$, so we have an injective induced map $\varepsilon : P^\vee/Q^\vee \to G$. This map lands in the center since elements of $P^\vee$ give integer inner products with roots, and hence elements $\varepsilon(x)$, $x \in P^\vee/Q^\vee$ act by the same scalar on all weight subspaces of any irreducible finite dimensional G-module. Reversing this argument, we see that this map is also surjective, so it is an isomorphism.

This implies that $\mathbb{W}$ acts trivially on Π. Indeed, the action of $\mathbb{W}$ on Π is induced by the action on the maximal torus $T \subset G$ of the normalizer $NT \subset G$ of this torus by conjugation. So the elements of Π are invariant under this action because they are central in G.

Thus, the subgroup $\mathbb{W}^a$ is normal in $\mathbb{W}^b$, and the quotient $\mathbb{W}^b/\mathbb{W}^a$ is naturally identified with Π (the identification is induced by the embedding $P^\vee \to \mathbb{W}^b$).

It is useful to introduce the notion of the length of an element of $\mathbb{W}^b$. By the definition, let the length of $w \in \mathbb{W}^b$, denoted by $l(w)$, be the number of positive roots which are mapped under w to negative roots. It is obvious that this number is finite, and that simple reflections have length 1. It is known that the length of an element of $\mathbb{W}^a$ given by a reduced decomposition with n factors is n, and that if $\lambda, \mu \in P^\vee$ are dominant then $l(\lambda+\mu) = l(\lambda) + l(\mu)$ (see e.g. [Ch1, Ch2], and references therein; the statements follow by looking at how λ and μ act on positive roots).

Let $\tilde{\Pi} \subset \mathbb{W}^b$ be the group of transformations that have length 0, i.e. those which map the sets of positive and negative affine roots to themselves.

It is clear that any element $w \in \mathbb{W}^b$ of length $n > 0$ can be represented as a product $w = \sigma w'$, where w' has length $n-1$. Indeed, let α be a simple positive root such that $w^{-1}\alpha$ is negative (clearly such exists, otherwise $w \in \tilde{\Pi}$). If β and $w\beta$ are positive roots, then $w\beta \neq \alpha$, so $s_\alpha w \beta$ is positive. In addition, $s_\alpha w(-w^{-1}\alpha) = \alpha$ is positive, so $l(s_\alpha w) = n-1$.

Thus, we get a factorization $\mathbb{W}^b = \tilde{\Pi}\mathbb{W}^a$. Moreover, the factorization of an element of $\mathbb{W}_b$ into a product of elements of $\mathbb{W}_a$ and $\tilde{\Pi}$ is unique, since $\mathbb{W}^a$ and $\tilde{\Pi}$ intersect trivially (as nontrivial elements of $\mathbb{W}^a$ have positive length). In other words, the exact sequence

$$1 \to \mathbb{W}^a \to \mathbb{W}^b \to \Pi \to 1$$

is split (canonically!), and the subgroup $\mathbb{W}^a \subset \mathbb{W}^b$ is complemented by the subgroup $\tilde{\Pi}$. Thus, we have $\mathbb{W}^b = \tilde{\Pi} \ltimes \mathbb{W}^a$.

In fact, the canonical splitting homomorphism $\eta : \Pi = P^\vee/Q^\vee \to \mathbb{W}^b$ can be constructed explicitly as follows.

For any $i = 1, ..., r$, let w_0^i be the maximal element of the Weyl group of the Levi subalgebra of $\mathfrak{g}$, whose Dynkin diagram is obtained from that of $\mathfrak{g}$ by throwing away the i-th vertex of the Dynkin diagram. Let $w_{[i]} = w_0 w_0^i$.

Recall that the fundamental coweights for $\mathfrak{g}$, $\omega_i^\vee$, $i = 1, ..., r$, are the elements of $P^\vee$ defined by $\alpha_j(\omega_i^\vee) = \delta_{ij}$. We say that a fundamental coweight $\omega_i^\vee$ is minuscule if $\theta(\omega_i^\vee) = 1$. (For non-minuscule coweights, this number is greater than 1).

It is known that for any element $\pi \neq 1$ of Π, there exists a unique minuscule fundamental coweight $\omega_i^\vee \in P^\vee$ which represents π in $P^\vee/Q^\vee$, and all minuscule fundamental coweights are obtained in this way exactly once.

Let us denote the coweight $\omega_i^\vee$ corresponding to π by $\omega^\vee(\pi)$, and the index i by i_π.

Proposition 50. *The homomorphism η is defined by $\eta(\pi) = t_{\omega^\vee(\pi)} w_{[i_\pi]}^{-1}$.*

Proof. We need to show that η lands in $\tilde{\Pi}$ and that it is a homomorphism. Let us prove the first statement. So let $\pi \in \Pi$, $i_\pi = i$, and let us show that $\eta(\pi) \in \tilde{\Pi}$.

It is clear that if $j \neq 0, i^*$ (where i^* is the dual vertex to i) then α_j is mapped under $\eta(\pi)$ to $\alpha(j')$, with $j' \neq 0$. Thus, we need to show that α_{i^*} and α_0 are also mapped to simple positive roots. A simple computation shows that this property is equivalent to the identity $w_0^i \alpha_i = \theta$. So let us prove this.

Let $w_0^i \alpha_i = \beta$. Clearly, $(\beta, \omega_i^\vee) = 1$. So β is a positive root of the form $\beta = \alpha_i + \gamma$, where γ is a linear combination of positive roots except α_i. Similarly, $\theta = \alpha_i + \gamma'$, and $\gamma' \geq \gamma$. Thus,

$$w_0^i \theta = w_0^i \alpha_i + w_0^i \gamma' = \alpha_i + \gamma + w_0^i \gamma'$$

Now, we see that since $\gamma' \geq \gamma$, the height of the right hand side (i.e. the sum of the multiplicities of the simple roots) is ≤ 1, and the equality is possible only if $\gamma = \gamma'$. But the right hand side is a positive root, so $\gamma' = \gamma$ and hence $\beta = \theta$.

Now we prove the second statement (that η is a homomorphism). Since η lands in $\tilde{\Pi}$, it is sufficient to check that the map $\Pi \to \mathbb{W}^b/\mathbb{W}^a = P^\vee/Q^\vee$ induced by η is a homomorphism. But this is obvious from the definition. □

Thus, Π can be identified with $\tilde{\Pi}$ via η, and can thus be regarded as a subgroup of $\mathbb{W}^b$; we will assume from now on that we have performed this identification. In particular, any element $\pi \in \Pi$ acts on $\mathbb{W}^a$ by conjugation. It is easy to show that this action permutes simple reflections according to an automorphism of the extended Dynkin diagram Γ^a of $\mathfrak{g}$. In other words, we have a homomorphism (in fact, an embedding) $\Pi \to \mathrm{Aut}(\Gamma^a)$.

Remark. It is easy to check that $\mathrm{Aut}(\Gamma^a) = \mathrm{Aut}(\Gamma) \ltimes \Pi$, where Γ is the Dynkin diagram of $\mathfrak{g}$.

Example. Consider $\mathfrak{g} = \mathfrak{s}l_2$. In this case the group $\mathbb{W}^a$ is generated by two elements s_0, s_1 such that $s_0^2 = s_1^2 = 1$, with no other relations. So we can think of $\mathbb{W}^a$ as

the group of all affine linear transformations of the integers, which is generated by $s_0(x) = -x$, $s_1(x) = 1 - x$. The group $\mathbb{W}^b$ is in this case the set of all affine linear transformations of the half-integers, so it has the form $\mathbb{W}^b = \Pi \ltimes \mathbb{W}^a$, where $\Pi = \{1, \pi\}$, $\pi(x) = \frac{1}{2} - x$. We see that $Q^\vee = \mathbb{Z}, P^\vee = \frac{1}{2}\mathbb{Z}$, and the action of Π on $\mathbb{W}^a$ is given by $\pi s_0 \pi^{-1} = s_1, \pi s_1 \pi^{-1} = s_0$, as predicted by the general theory. We have $\omega_1^\vee(x) = x + 1/2$, and the element $w_{[1]}$ is given by $w_{[1]}(x) = -x$.

8.3. **Intertwining operators and expectation values.** Let X be a module over $\tilde{\mathfrak{h}}$ which has a weight decomposition, and let M be a module over $U_q(\tilde{\mathfrak{g}})$ from category $\mathcal{O}$. For any weight $\tilde{\nu}$, define the space $(M\hat{\otimes}X)[\tilde{\nu}]$ to be $\hat{\oplus}_{\tilde{\beta}} M[\tilde{\beta}] \otimes X[\tilde{\nu} - \tilde{\beta}]$ (where $\hat{\oplus}$ is the completed direct sum, i.e. the Cartesian product over all $\tilde{\beta}$). Elements of this space are arbitrary (possibly infinite) sums of tensors whose first and second components are homogeneous. Define the completed tensor product $M\hat{\otimes}X$ to be $\oplus_{\tilde{\nu}}(M\hat{\otimes}X)[\tilde{\nu}]$ (an algebraic direct sum).

Let $M_{\tilde{\lambda}}$ be the Verma module over $U_q(\tilde{\mathfrak{g}})$ with highest weight $\tilde{\lambda}$. Let $\bar{V}$ be a finite dimensional representation of $U_q(\hat{\mathfrak{g}})$, and V the corresponding loop representation. Consider an intertwining operator $\Phi : M_{\tilde{\lambda}} \to M_{\tilde{\mu}}\hat{\otimes}V$. Like for intertwiners into the usual tensor product, we have $\Phi v_{\tilde{\lambda}} = v_{\tilde{\mu}} \otimes v + ...$, where ... denote terms of lower weight in the first component, and $v \in V[\tilde{\lambda} - \tilde{\mu}]$ (but now the sum denoted by ... may be infinite). By analogy with the previous setting, we will call v the expectation value of Φ and write $<\Phi>= v$.

Let us now generalize Lemma 1 to the affine case. For clarity we will split this generalization into two lemmas.

Let V be a loop representation, and $\tilde{\nu}$ a weight in V.

Lemma 51. *For generic $\tilde{\lambda}$ the map $\Phi \to <\Phi>$ is an isomorphism of vector spaces $Hom_{U_q(\tilde{\mathfrak{g}})}(M_{\tilde{\lambda}}, M_{\tilde{\lambda}-\tilde{\nu}}\hat{\otimes}V) \to V[\tilde{\nu}]$.*

Remark 1. Here "generic" means away from a countable (possibly infinite) number of hyperplanes.

Remark 2. Note that the lemma would be wrong if we used $\otimes$ instead of $\hat{\otimes}$.

Remark 3. Note that in Lemma 51, the central charges of $\tilde{\lambda}$ and $\tilde{\lambda} - \tilde{\nu}$ are the same, by Proposition 48, unless the spaces are zero.

Proof. Straightforward, as in Lemma 1; see also Theorem 3.1.1 in [EFK]. □

This lemma allows one to define the interwining operator $\Phi^v_{\tilde{\lambda}}$ with expectation value v. As before, it has coefficients which are rational functions of $\tilde{\lambda}$ or $q^{(\tilde{\lambda},\alpha_i)}$.

Lemma 52. *The map $\Phi \to <\Phi>$ is an isomorphism for dominant weights $\tilde{\lambda}$ with sufficiently large coordinates $\tilde{\lambda}(h_i)$.*

Proof. The lemma is proved by arguments similar to those in [ESt]. Namely, similarly to [ESt], one can write down an explicit formula for $\Phi^v_{\tilde{\lambda}} v_{\tilde{\lambda}}$, and show that its poles are

all of first order and can occur only on hyperplanes $(\tilde{\lambda}+\tilde{\rho},\alpha)=\frac{n}{2}(\alpha,\alpha)$ for positive roots α and such $n>0$ that $V[\tilde{\nu}+n\alpha]\neq 0$. If a dominant weight $\tilde{\lambda}$ belongs to such a hyperplane, then $(\alpha,\alpha)>0$, so α is a real root. But it is clear that there exists a number N such that for $n\geq N$ one has $V[\tilde{\nu}+n\alpha]=0$ for any weight $\tilde{\nu}$ of V and any real root α. □

Remark. Note that the last statement of the proof of Lemma 52 would be false for imaginary roots.

8.4. **The dynamical Weyl group for loop representations.** It is easy to see that loop representations are locally finite, so the dynamical Weyl group operators $A_{w,V}(\tilde{\lambda})$, $w\in\mathbb{W}^a$, are already defined on them. It is easy to see that these operators are linear over $\mathbb{C}[z,z^{-1}]$.

Moreover, we have an analog of Proposition 15:

Proposition 53. *One has*

$$\Phi v^{\tilde{\lambda}}_{w\cdot\tilde{\lambda}}=v^{\tilde{\mu}}_{w\cdot\tilde{\mu}}\otimes A_{w,V}(\tilde{\lambda})<\Phi>+\textit{lower weight terms.}$$

Proof. The proof is analogous to the proof of Propsition 15. □

8.5. **Fusion matrices.** Now let us generalize to the affine case the construction of fusion matrices.

First of all, define completed tensor products of Laurent polynomial spaces. Let $\bar{V}_i$ be vector spaces, and $V_i=\bar{V}_i[z,z^{-1}]$. Define $V_1\overrightarrow{\otimes}...\overrightarrow{\otimes}V_N$ to be

$$V_1\overrightarrow{\otimes}...\overrightarrow{\otimes}V_N:=(\bar{V}_1\otimes...\otimes\bar{V}_n)[[z_2/z_1,...,z_N/z_{N-1}]][z_1,z_1^{-1}],$$

where z_i denote the formal parameters corresponding to V_i. It is clear that if V_i are loop representations of $U_q(\tilde{\mathfrak{g}})$ then $V_1\overrightarrow{\otimes}...\overrightarrow{\otimes}V_N$ is also a representation of this algebra, which is locally finite. In fact, $V_1\overrightarrow{\otimes}...\overrightarrow{\otimes}V_N$ is a certain completion of the ordinary tensor product $V_1\otimes...\otimes V_N$.

Now let $\tilde{\lambda}\in\tilde{\mathfrak{h}}^*$ be a generic weight. Let V,U be loop representations of $U_q(\tilde{\mathfrak{g}})$, and $v\in V[\tilde{\mu}]$, $u\in U[\tilde{\nu}]$.

Consider the composition

$$\Phi^{u,v}_{\tilde{\lambda}}:\ M_{\tilde{\lambda}}\xrightarrow{\Phi^v_{\tilde{\lambda}}}M_{\tilde{\lambda}-\tilde{\mu}}\hat{\otimes}V\xrightarrow{\Phi^u_{\tilde{\lambda}-\tilde{\mu}}\otimes 1}M_{\tilde{\lambda}-\tilde{\mu}-\tilde{\nu}}\hat{\otimes}(U\overrightarrow{\otimes}V).$$

(It is easy to see that this compoosition is well defined; see also [EFK] for explanations). Then $\Phi^{u,v}_{\tilde{\lambda}}\in\mathrm{Hom}_{U_q(\mathfrak{g})}(M_{\tilde{\lambda}},M_{\tilde{\lambda}-\tilde{\mu}-\tilde{\nu}}\hat{\otimes}(U\overrightarrow{\otimes}V))$. Let $x=<\Phi^{u,v}_{\tilde{\lambda}}>$. Since U,V have a weight decomposition by Proposition 48, the assignment $(u,v)\mapsto x$ is bilinear, and naturally extends to a zero weight map

$$J_{UV}(\tilde{\lambda}):\ U\overrightarrow{\otimes}V\to U\overrightarrow{\otimes}V,$$

linear over $\mathbb{C}[[z_2/z_1]][z_1,z_1^{-1}]$. This means, the operator $J_{UV}(\tilde{\lambda})$ can be understood as an element of $\mathrm{End}(\bar{U}\otimes\bar{V})[[z_2/z_1]]$.

The operator $J_{UV}(\tilde\lambda)$ is called the fusion matrix of U and V. The fusion matrix $J_{UV}(\tilde\lambda)$ is a power series in $z = z_2/z_1$ of the form

$$J_{UV}(\tilde\lambda)(z) = \sum_{n\geq 0} J_{UV,n}(\tilde\lambda) z^n$$

where $J_{UV,n}(\tilde\lambda)$ is a rational function of $\tilde\lambda$ (respectively $q^{\tilde\lambda}$). Also, $J_{UV,0}(\tilde\lambda) = J_{\bar U\bar V}(\lambda)$, where λ is the $\mathfrak{h}^*$-part of $\tilde\lambda$, where $\bar U, \bar V$ are considered as $U_q(\mathfrak{g})$-modules. In particular, this shows that $J_{UV}(\tilde\lambda)$ is invertible.

Let us also define the multicomponent fusion matrix. Namely, let $V_1, ..., V_N$ be loop representations of $U_q(\tilde{\mathfrak{g}})$, Then define an operator

$$J^{1..N}(\tilde\lambda):\ V_1 \overrightarrow{\otimes} ... \overrightarrow{\otimes} V_N \to V_1 \overrightarrow{\otimes} ... \overrightarrow{\otimes} V_N,$$

by

$$J^{1..N}(\tilde\lambda) := J^{1,2...N}_{V_1,V_2\otimes...\otimes V_N}(\tilde\lambda)...J^{N-1,N}_{V_{N-1},V_N}(\tilde\lambda).$$

This operator can be regarded as an element of $\mathrm{End}(\bar V_1 \otimes ... \otimes \bar V_N)[[z_2/z_1, ..., z_N/z_{N-1}]]$.

Remark. It is easy to see that the matrix elements of the operator $J^{1...N}$ are the expectation values of products of intertwining operators (i.e. the correlation functions for the Wess-Zumino-Witten conformal field theory, for $q = 1$, and its q-deformation, for $q \neq 1$), which are the main objects of discussion in [EFK]. In particular, it is known that if the representations $\bar V_i$ are irreducible then the formal series $J^{1...N}(\tilde\lambda, \zeta_1, ..., \zeta_{N-1})$ is convergent (for a generic $\tilde\lambda$) to an analytic function of ζ_i in some neighborhood of zero (see [EFK] and references therein). However, we will not need this fact in this paper.

8.6. **The multicomponent dynamical action.** Let $V_1, ..., V_N$ be loop representations of $U_q(\tilde{\mathfrak{g}})$. The multicomponent shifted dynamical action $w\bullet$ of $\tilde{\mathbb{W}}^a$ on $V_1 \overrightarrow{\otimes} ... \overrightarrow{\otimes} V_N$ is defined in the same way as it was defined in the general Kac-Moody case.

Similarly to the general Kac-Moody case, we have the following proposition.

Proposition 54. *The operator $J^{1...N}$ conjugates the shifted dynamical action of $\tilde{\mathbb{W}}^a$ into its shifted multicomponent dynamical action. That is,*

$$J^{1...N}(w\bullet) = (w\circ)J^{1...N}.$$

Proof. The proof is the same as that of Lemma 18 and Lemma 46. □

Define

$$\mathcal{J}^{1...N}(\tilde\lambda) = J^{1...N}(-\tilde\lambda - \tilde\rho + \frac{1}{2}\sum_{i=1}^{N} h^{(i)})$$

Corollary 55. *The operator $\mathcal{J}^{1...N}$ conjugates the (unshifted) dynamical action of $\tilde{\mathbb{W}}^a$ into its multicomponent dynamical action. That is,*

$$\mathcal{J}^{1...N}(w\diamond) = (w*)\mathcal{J}^{1...N}.$$

Proof. Clear. □

8.7. **Trigonometric KZ equations for $\mathcal{J}^{1...N}$ $(q=1)$.** In this section we will assume that $\tilde\lambda=(\lambda,k,0)$. Let $(,)$ be the inner product on $\mathfrak{g}$ that we defined before, and let e_α, f_α be Cartan-Weyl generators such that $(e_\alpha,f_\alpha)=1$.

Define the Drinfeld r-matrix for $\mathfrak{g}$

$$r=\sum_{\alpha>0}e_\alpha\otimes f_\alpha+\frac{1}{2}\sum_i x_i\otimes x_i$$

(recall that x_i is an orthonormal basis of $\mathfrak{h}$).

Let $V_1,...,V_N$ be loop representations of $U(\tilde{\mathfrak{g}})$. Define the trigonometric KZ operators on $V_1\overrightarrow{\otimes}...\overrightarrow{\otimes}V_N$:

$$\nabla_i(\tilde\lambda)=mkz_i\frac{\partial}{\partial z_i}+\sum_{j\neq i}\frac{z_ir_{ji}+z_jr_{ij}}{z_i-z_j}+\lambda^{(i)},$$

where the rational functions of z_i on the right hand side are expanded in a power series with respect to z_i/z_{i-1}.

Remark 1. Observe that $mkz_i\frac{\partial}{\partial z_i}+\lambda^{(i)}=\tilde\lambda^{(i)}$ on $V_1\overrightarrow{\otimes}V_2\overrightarrow{\otimes}...\overrightarrow{\otimes}V_N$, and that the operator $\frac{z_ir_{ji}+z_jr_{ij}}{z_i-z_j}$ can be thought of as the action of the Drinfeld r-matrix of the *affine* algebra $\tilde{\mathfrak{g}}$ in the tensor product of two loop representations.

Remark 2. We note that our trigonometric KZ operators differ from those of [TV] by a change: $\lambda\to-\lambda$, $\kappa\to-\kappa$, $z_i\to z_i^{-1}$ (apart from the overall factor of m). This is the cause of a number of sign discrepancies between [TV] and this paper.

Let $\nabla_i^0(\tilde\lambda)$ be the "Cartan" part of $\nabla_i(\tilde\lambda)$, i.e.

$$\nabla_i^0(\tilde\lambda)=mkz_i\frac{\partial}{\partial z_i}+\sum_{j<i}\sum_{l=1}^r x_l^{(i)}\otimes x_l^{(j)}-\sum_{j>i}\sum_{l=1}^r x_l^{(i)}\otimes x_l^{(j)}+\lambda^{(i)}.$$

Theorem 56. [TK, FR] *(trigonometric KZ equations) One has*

$$\nabla_i(\tilde\lambda)\mathcal{J}^{1...N}(\tilde\lambda)=\mathcal{J}^{1...N}(\tilde\lambda)\nabla_i^0(\tilde\lambda).$$

Proof. This is, after some transformations, the content of Theorem 3.8.1 in [EFK]. This is also the multicomponent version of the ABRR equation for affine Lie algebras, projected to the product of loop representations (see [ES]). □

Remark 1. Note that in Theorem 3.8.1 of [EFK], there is a misprint: there should be a factor $\frac{1}{2}$ in front of $<\mu_i,\mu_i+2\rho>$.

Remark 2. In the KZ equation for conformal blocks, the central charge k occurs in a combination $k+h^\vee$ (see [EFK]). This shift of k by $h^\vee$ is absent here because when passing from J to $\mathcal{J}$, we have performed a shift by $\tilde\rho$, which, in particular, involves a shift of k by $h^\vee$.

Remark 3. We note that the finite-dimensional analog of Theorem 56 (i.e. the corresponding statement for $\mathfrak{g}$ and not for $\tilde{\mathfrak{g}}$) appears in [TV] as formula (2) (in the case $N=2$). We also note that formula (2) of [TV] can be generalized to an arbitrary Kac-Moody algebra.

Theorem 57. *The trigonometric KZ operators $\nabla_i(\tilde\lambda)$ commute with the dynamical action of $\tilde{\mathbb{W}}^a$.*

Proof. We have seen that the operator $\mathcal{J}^{1...N}$ conjugates the operators ∇_i to the diagonal operators ∇_i^0, and the dynamical action of the braid group to the multicomponent dynamical action. It is easy to see that the multicomponent dynamical action commutes with the operators ∇_i^0. This implies the desired statement. ☐

8.8. **Trigonometric qKZ equations for $\mathcal{J}^{1...N}$ ($q \neq 1$).** Let us now describe the generalization of the content of the previous section to the quantum case ($q \neq 1$).

Let $V_1, ..., V_N$ be loop representations of $U_q(\tilde{\mathfrak{g}})$. Let $\mathcal{R}_{ij}(z_i/z_j) \in \mathrm{End}(\bar V_i \otimes \bar V_j)[[z_i/z_j]]$ be the projection of the universal R-matrix.

Remark 1. This projection is well defined. Indeed, since $c = 0$ in V_i and V_j by Proposition 48, the part $q^{m(c\otimes d+d\otimes c)}$ of the universal R-matrix disappears when it is evaluated on $V_i \otimes V_j$; thus the R-matrix defines an element of $\mathrm{End}(\bar V_i \otimes \bar V_j)[[z_i/z_j]]$.

Remark 2. We note (see [FR],[EFK]) that if the representations $\bar V_1, \bar V_2$ are irreducible then the R-matrix $\mathcal{R}_{12}(z_1/z_2) \in \mathrm{End}(\bar V_1 \otimes \bar V_2)[[z_1/z_2]]$ is not only a power series, but is actually an analytic function (for small z_1/z_2), which moreover is a product of a scalar meromorphic (transcendental) function on $\mathbb{C}$ (which is holomorphic at 0) and an operator-valued rational function.

Let $p = q^{2mk}$.

Define the trigonometric qKZ operators

$$\nabla_i^q(\tilde\lambda) = \mathcal{R}_{i+1,i}(\frac{z_{i+1}}{z_i})...\mathcal{R}_{Ni}(\frac{z_N}{z_i})(q^\lambda)_i T_{i,p}\mathcal{R}_{i1}(\frac{z_i}{z_1})^{-1}...\mathcal{R}_{i,i-1}(\frac{z_i}{z_{i-1}})^{-1},$$

where $T_{i,p}z_j = z_j p^{\delta_{ij}}$.

Let $\nabla_i^{q,0}$ be the "Cartan part" of ∇_i^q. That is,

$$\nabla_i^{q,0}(\tilde\lambda) = q^{\sum_l x_l^i \otimes (\sum_{j>i} x_l^{(j)} - \sum_{j<i} x_l^{(j)})}(q^\lambda)_i T_{i,p}.$$

Theorem 58. [FR] *(trigonometric qKZ equations) One has*

$$\nabla_i^q(\tilde\lambda)\mathcal{J}^{1...N}(\tilde\lambda) = \mathcal{J}^{1...N}(\tilde\lambda)\nabla_i^{q,0}(\tilde\lambda).$$

Proof. This is the main result of [FR]; see also Theorem 10.3.1 in [EFK] (where the simply laced case is treated). This is also the multicomponent version of the ABRR equation for quantum affine algebras, projected to the product of loop representations (see [ES]). ☐

Remark. Note that in the non-simply-laced case, the statement of the main theorem of [FR] should be corrected. Namely, the quantity $k + h^\vee$ in the quantum KZ equations should be replaced by $m(k + h^\vee)$, which agrees with our statements here.

Theorem 59. *The trigonometric qKZ operators $\nabla_i^q(\tilde\lambda)$ commute with the dynamical action of $\tilde{\mathbb{W}}^a$.*

Proof. Analogous to Theorem 57. ☐

9. The dynamical difference equations

In this section we would like to apply the material of the previous section to deriving the dynamical difference equations from [TV]. Before we do so, we need to establish a few auxiliary results about evaluation representations.

9.1. **The operators $\mathcal{A}_i(\mathbf{z},\lambda)$.** Let V be a loop representation of $U_q(\tilde{\mathfrak{g}})$. Let D be the common denominator of $(\omega_i,\omega_j^\vee)$, where ω_i are the fundamental weights of $\mathfrak{g}$. Let $V^e=V\otimes_{\mathbb{C}[z,z^{-1}]}\mathbb{C}[z^{1/D},z^{-1/D}]$. The space V^e has a natural structure of a representation of $U_q(\tilde{\mathfrak{g}})$. We call V^e the extended version of V.

Let $\pi_i\in\Pi$, $w_i\in\mathbb{W}^a$ be the elements such that $t_{\omega_i^\vee}=\pi_iw_i$. For example, if $\omega_i^\vee=\omega^\vee(\pi)$ is the minuscule weight corresponding to $\pi\in\Pi$, then $\pi_i=\pi, w_i=w_{[i]}$.

Let $q=1$. For $i=1,...,r$, and any loop representation V of $\tilde{\mathfrak{g}}$, consider the operators $\Pi_{i,V}$ on V^e given by the formula

$$\Pi_{i,V}=z^{-\omega_i^\vee}(A^+_{w_i,V})^{-1}.$$

Also, let $\hat\pi_i$ denote the automorphism of $U_q(\tilde{\mathfrak{g}})$ defined by permuting the labels of the generators according to the permutation $\pi_i\in\mathrm{Aut}(\Gamma^a)$.

Lemma 60. *There is a unique Lie algebra automorphism ξ_i of $\tilde{\mathfrak{g}}$, which satisfies the equation*

$$\xi_i(a)|_{V^e}=\Pi_{i,V}a|_{V^e}\Pi_{i,V}^{-1},\ a\in\tilde{\mathfrak{g}}$$

in all loop representations.

(ii) One has

$$\xi_i(e_j)=c_{ij}e_{\pi_i(j)},\ \xi_i(f_j)=c_{ij}^{-1}f_{\pi_i(j)},\ \xi_i(h_j)=h_{\pi_i(j)},\xi_i(\partial)=\partial,$$

where c_{ij} are nonzero complex numbers, and ∂ is a principal grading element.

(iii) There exist elements $x_i\in\tilde{\mathfrak{h}}$ such that $\xi_i=\hat\pi_i\circ Ad(e^{x_i})$.

Proof. The proof is easy. ☐

So, let us define the Hopf algebra automorphisms ξ_i of $U_q(\tilde{\mathfrak{g}})$ for any q, using the same formulas for the action of ξ_i on generators. It is easy to see that part (iii) of Lemma 60 is valid in the q-case, with the same elements x_i.

Remark. We note that for $q\neq1$ one no longer has $\xi_i=\mathrm{Ad}(z^{-\omega_i^\vee}(A^+_{w_i,V})^{-1})$.

Now let $q\neq1$, and let V be a loop representation corresponding to an evaluation representation $\bar V$.

Proposition 61. *There exists a unique operator $\Pi^q_{i,V}$ on V, which is a q-deformation of the classical operator $\Pi_{i,V}$, such that*

$$\xi_i(a)|_{V^e}=\Pi^q_{i,V}a|_{V^e}(\Pi^q_{i,V})^{-1},$$

and the determinant of $\Pi^q_{i,V}$ on $V^e/(z^{1/D}-1)$ is independent on q. More specifically, the first condition defines this operator uniquely up to a constant, while the second condition fixes the constant.

Proof. Recall that any automorphism of an algebra acts on the set of equivalence classes of representations of this algebra. All we need to show is that the representation V is stable under the automorphism ξ_i for any q.

It follows from Drinfeld's highest weight theory of finite dimensional representations of $U_q(\hat{\mathfrak{g}})$ that there is a unique, up to a shift of parameter, finite dimensional representation of $U_q(\hat{\mathfrak{g}})$ with the same $U_q(\mathfrak{g})$-character as $\bar{V}$ (namely, all such representations have the form $\bar{V}(a)$ for some a). On the other hand, it is easy to check that ξ_i does not change the $U_q(\mathfrak{g})$-character of a representation. This implies the statement. $\square$

Consider now the completed tensor product

$$V = V_1^e \overrightarrow{\otimes} V_2^e \overrightarrow{\otimes} ... \overrightarrow{\otimes} V_N^e := (V_1 \overrightarrow{\otimes} V_2 \overrightarrow{\otimes} ... \overrightarrow{\otimes} V_N) \otimes_{\mathbb{C}[z_j^{\pm 1}]} \mathbb{C}[z_j^{\pm 1/D}]$$

of extended loop representations associated to evaluation representations $\bar{V}_i$. Define the operators $\Pi_{i,V}$ on V^e by the formula

$$\Pi_{i,V}^q = \Pi_{i,V_1}^q \otimes ... \otimes \Pi_{i,V_N}^q.$$

Let $\mathbf{z} = (z_1, ..., z_N)$. Consider the following operators on V:

$$\mathcal{A}_i(\mathbf{z}, \tilde{\lambda}) = \Pi_{i,V}^q \mathcal{A}_{w_i,V}(\tilde{\lambda}).$$

It is easy to see that the operators $\mathcal{A}_i(\mathbf{z}, \tilde{\lambda})$ really depend only of the components λ, k of $\tilde{\lambda}$. Also, the parameter k will be fixed in the following discussion, so we will not write the dependence on it explicitly. Thus we will denote $\mathcal{A}_i(\mathbf{z}, \tilde{\lambda})$ simply by $\mathcal{A}_i(\mathbf{z}, \lambda)$.

Remark. It is easy to see that the matrix elements of the operator $\mathcal{A}_i(\mathbf{z}, \lambda)$ are Laurent polynomials of $z_j^{1/D}$ with coefficients in rational functions of λ (or q^λ).

Let $\kappa = mk$. Our main result in this subsection is the following.

Theorem 62. *The operators $\mathcal{A}_i(\mathbf{z}, \tilde{\lambda})$ form a holonomic system. That is,*

$$\mathcal{A}_i(\mathbf{z}, \lambda + \kappa\omega_j^\vee)\mathcal{A}_j(\mathbf{z}, \lambda) = \mathcal{A}_j(\mathbf{z}, \lambda + \kappa\omega_i^\vee)\mathcal{A}_i(\mathbf{z}, \lambda).$$

Proof. We have (dropping V from the subscripts and q from the superscripts for brevity):

$$\mathcal{A}_i(\mathbf{z}, \lambda + \kappa\omega_j^\vee)\mathcal{A}_j(\mathbf{z}, \lambda) = \Pi_i \mathcal{A}_{w_i}(t_{\omega_j^\vee}\tilde{\lambda})\Pi_j \mathcal{A}_{w_j}(\tilde{\lambda}) = \Pi_i \Pi_j \xi_j^{-1}(\mathcal{A}_{w_i}(t_{\omega_j^\vee}\tilde{\lambda}))\mathcal{A}_{w_j}(\tilde{\lambda}) =$$

$$= \Pi_i \Pi_j e^{-x_j} \hat{\pi}_j^{-1}(\mathcal{A}_{w_i}(t_{\omega_j^\vee}\tilde{\lambda}))e^{x_j}\mathcal{A}_{w_j}(\tilde{\lambda}) = \Pi_i \Pi_j e^{-x_j + \pi_j^{-1}(w_i)(x_j)} \hat{\pi}_j^{-1}(\mathcal{A}_{w_i}(t_{\omega_j^\vee}\lambda))\mathcal{A}_{w_j}(\tilde{\lambda}) =$$

$$\Pi_i \Pi_j e^{-x_j + \pi_j^{-1}(w_i)(x_j)} \mathcal{A}_{\pi_j^{-1}(w_i)}(w_j\tilde{\lambda})\mathcal{A}_{w_j}(\tilde{\lambda}).$$

Now recall that in the braid group $\tilde{\mathbb{W}}^a$ we have $t_{\omega_i^\vee} t_{\omega_j^\vee} = t_{\omega_j^\vee} t_{\omega_i^\vee}$, and hence $\pi_j^{-1}(w_i)w_j = \pi_i^{-1}(w_j)w_i$ (with the length of both being $l(w_i) + l(w_j)$ in the affine Weyl group). This implies that the product $\mathcal{A}_{\pi_j^{-1}(w_i)}(w_j\tilde{\lambda})\mathcal{A}_{w_j}(\tilde{\lambda})$ is symmetric under interchanging i and j (the 1-cocycle relation). Thus, the theorem is equivalent to the statement that the expression $G_{ij} = \Pi_i \Pi_j e^{-x_j + \pi_j^{-1}(w_i)(x_j)}$ is symmetric in i and j. But this statement is $\tilde{\lambda}$-independent, so it is sufficient to prove the theorem in the limit $\tilde{\lambda} \to \infty$ (respectively, $q^{(\tilde{\lambda}, \alpha_l)} \to +\infty$). We can also assume that V is a single loop representation.

Now, for $q = 1$, this statement is clear since the operators $\mathcal{A}_i$ in the limit $\tilde{\lambda} \to \infty$ are just $z^{-\omega_i^\vee}$. In particular, for $q = 1$, conjugation by G_{ij} and conjugation by G_{ji} act in the same way on the generators of $\tilde{\mathfrak{g}}$. But since x_i are q-independent, this action is independent on q. Thus, the two actions coincide even at $q \neq 1$. Since V is an irreducible module over $U_q(\tilde{\mathfrak{g}})$, by Schur's lemma this means that $G_{ij}G_{ji}^{-1} = C_{ij}(q)$, where $C_{ij}(q)$ are nonzero complex numbers, and $C_{ij}(1) = 1$.

Finally, taking the determinants, we find that some power of C_{ij} is 1, so by continuity $C_{ij} = 1$ also. The theorem is proved. □

9.2. **The dynamical difference equations.** Theorem 62 implies the following result, which applies both to the classical and the quantum situation.

Theorem 63. *Consider the system of difference equations*

$$\varphi(\mathbf{z}, \lambda + \kappa\omega_i^\vee) = \mathcal{A}_i(\mathbf{z}, \lambda)\varphi(\mathbf{z}, \lambda),\ \ i = 1, ..., r.$$

with respect to a function φ of $\lambda \in \mathfrak{h}^*$ and $z_1, ..., z_N$ with values in $\bar{V}_1 \otimes ... \otimes \bar{V}_N$. Then:

(i) This system is consistent, (i.e. $\mathcal{A}_j(\mathbf{z}, \lambda + \kappa\omega_i^\vee)\mathcal{A}_i(\mathbf{z}, \lambda) = \mathcal{A}_i(\mathbf{z}, \lambda + \kappa\omega_j^\vee)\mathcal{A}_j(\mathbf{z}, \lambda)$ for all i, j).

(ii) This system commutes with the KZ (qKZ) equations:

$$\mathcal{A}_i(\mathbf{z}, \lambda)\nabla_l^q(\lambda) = \nabla_l^q(\lambda + \kappa\omega_i^\vee)\mathcal{A}_i(\mathbf{z}, \lambda).$$

Proof. Statement (i) is exactly Theorem 62. Statement (ii) follows from the fact that the operator $\Pi_{i,V}$ commutes with the KZ (qKZ) equations. □

Definition. We call this system of difference equations *the dynamical difference equations* for $U_q(\tilde{\mathfrak{g}})$.

9.3. **The expression of the dynamical difference equations via the operators $B_w^+(\lambda)$ in the case $q = 1$.** For any quantized Kac-Moody algebra, let $\mathcal{B}_w^+(\lambda) := B_w^+(-\lambda-\rho)$. Let us write the dynamical difference equations from the previous section in terms of the operators $\mathcal{B}_w^+$ corresponding to the quantum affine algebra $U_q(\tilde{\mathfrak{g}})$, in the case $q = 1$.

Proposition 64. *Let $q = 1$. Then the linear operator $\mathcal{A}_i(\mathbf{z}, \lambda)$ is defined by the formula*

$$\mathcal{A}_i(\mathbf{z}, \lambda) = \left(\prod (z_j^{-\omega_i^\vee})^{(j)}\right) \mathcal{B}_{w_i}^+(\tilde{\lambda}),\ \ i = 1, ..., r$$

where $\tilde{\lambda} = (\lambda, k, 0)$. In particular, the dynamical difference equations for $q = 1$ have the form

$$\varphi(\mathbf{z}, \lambda + \kappa\omega_i^\vee) = \left(\prod (z_j^{-\omega_i^\vee})^{(j)}\right) \mathcal{B}_{w_i}^+(\tilde{\lambda})\varphi(\mathbf{z}, \lambda).$$

Proof. This is immediate from the previous results. □

From this formula, it is seen that (as was observed already in [TV]) the difference equations corresponding to *minuscule* fundamental coweights are especially simple. Namely, in this case, $w_i \in \mathbb{W} \subset \mathbb{W}^a$, which implies that the operator $\mathcal{B}^+_{w_i}(\tilde{\lambda})$ is independent of z_i and of κ.

Let us now write down an explicit formula for $\mathcal{B}^+_w(\tilde{\lambda})$, $w \in \mathbb{W}^a$, in a representation of the form $V_1^e \overrightarrow{\otimes} V_2^e \overrightarrow{\otimes} ... \overrightarrow{\otimes} V_N^e$, where V_i^e are extended loop representations.

For a real root α of $\tilde{\mathfrak{g}}$, let $\bar{\alpha}$ be its $\mathfrak{h}^*$-part, and let $m_\alpha = \alpha(d)$.

For a root β of $\mathfrak{g}$, let Z_β be the operator $z_1^{\beta^{\vee(1)}/2} ... z_N^{\beta^{\vee(N)}/2}$.

Proposition 65. *Let $\delta : w = s_{i_1} ... s_{i_l}$ be a reduced decomposition of w. Let α^j be the corresponding roots and $m_j := m_{\alpha^j}$. Then on the representation $V_1^e \overrightarrow{\otimes} V_2^e \overrightarrow{\otimes} ... \overrightarrow{\otimes} V_N^e$, one has*

$$\mathcal{B}^+_w(\tilde{\lambda}) = \prod_{j=1}^{l} \left(Z_{\bar{\alpha}^j}^{m_j} \cdot p(-\lambda(h_{\bar{\alpha}^j}) - \kappa m_j - 1, h_{\bar{\alpha}^j}, e_{\bar{\alpha}^j}, e_{-\bar{\alpha}^j}) \cdot Z_{\bar{\alpha}^j}^{-m_j} \right).$$

Proof. The proposition follows immediately from Proposition 31 and the definitions. □

Corollary 66. *The dynamical difference equations for $q = 1$ coincide with* [TV]*, eq. (16), after the change of variables $z_i \to z_i^{-1}$, $\kappa \to -\kappa$, $\lambda \to -\lambda$.*

Proof. The proof is by a straightforward comparison of the two systems. □

Corollary 67. *The dynamical difference equations (16) in* [TV] *are consistent and commute with the trigonometric KZ equations (in the form of* [TV]*).*

We note that the consistency of the dynamical difference equations was shown in [TV], Lemma 23, using Cherednik's theory of affinization of R-matrices. The compatibility of the dynamical difference equations with the trigonometric KZ equations was proved in [TV] for Lie algebras of type other than E_8, F_4, G_2 (Theorem 24), and was conjectured for the remaining three types (the proof of Theorem 24 uses the existence of a minuscule fundamental coweight, and hence fails for E_8, F_4, G_2). Proposition 67 implies that this conjecture is correct.

9.4. **The case $\mathfrak{g} = \mathfrak{sl}_n$, $q \neq 1$.** Consider the case $\mathfrak{g} = \mathfrak{sl}_n$, $q \neq 1$. To make our picture complete, let us calculate the operators $\Pi^q_{i,V}$ for evaluation representations of $U_q(\hat{\mathfrak{g}})$.

First of all, it is easy to check that the operators $\Pi^q_{i,V}$ are consistent with tensor product (i.e. they agree with morphisms mapping one evaluation representation into a product of others). On the other hand, it follows from Drinfeld's highest weight theory of finite dimensional representations that any evaluation representation occurs in a tensor product of shifted vector representations (see e.g. [CP]). Thus, it is sufficient to compute the operators $\Pi_{i,V}$ for the vector representation.

So let $\bar{V}$ be the vector representation of $U_q(\hat{\mathfrak{g}})$, evaluated at $z = 1$. Recall that this representation has a basis $v_1, ..., v_n$ in which the action of the generators is defined by

the following formulas:

$$e_i \to E_{i,i+1}, f_i \to E_{i+1,i}, h_i \to E_{i,i} - E_{i+1,i+1},$$

where E_{ij} is the elementary matrix, and the index i is understood as an element of $\mathbb{Z}/n\mathbb{Z}$.

The crucial properties of this formula is that these formulas are independent on q, and in particular are the same as those for $q = 1$. Therefore, we get

Proposition 68. *The matrices of the operators $\Pi^q_{i,V}$ in the basis $\{v_j\}$ are independent of q. They are given by the formula*

$$\Pi^q_{i,V} v_j = \gamma_{ij} v_{j+i},$$

where

$$\gamma_{ij} = z^{\frac{i}{n}-1}, \quad i+j > n,$$

and

$$\gamma_{ij} = (-z^{1/n})^i, \quad i+j \le n.$$

(where in the last two formulas i, j are integers, not elements of $\mathbb{Z}/n\mathbb{Z}$).

For example, the matrix of the operator $\Pi^q_{1,V}$ for $n = 2$ is

$$\Pi^q_{1,V} = \begin{pmatrix} 0 & -z^{-1/2} \\ z^{1/2} & 0 \end{pmatrix}.$$

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